A Nonlinear Piecewise Bicubic Interpolation Method for Cell-Averaged Data on Regular Grids

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Nonlinear Approximation for High-Dimensional Problems.
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Outline

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Introduction

Data

Given the averages $m_{i,j} = \frac{1}{h^2}\int_{x_{i-1}}^{x_i}\int_{y_{j-1}}^{y_j} F(x,y)dxdy$ on a rectangular mesh $x_0 = a < x_1 < \ldots < x_n = b$ and $y_0 = c < y_1 < \ldots < y_m = d$ of a function $F(x,y)$ defined in $[a,b]\times[c,d]$:

(we assume $n = m$, $a = c = 0$, $b = d = 1$, $h = 1/n$, $x_i = ih$, $y_j = jh$)

Goal

Find $H(x,y)$ s.t. $H(x,y) \approx F(x,y)$ that is smooth, accurate, and monotonic on R=$[0,1]\times[0,1]$

Piecewise Bicubic Hermite Interpolation

Given the values $\{f_{i,j}\}$, $\{f_{x,i,j}\}$, $\{f_{y,i,j}\}$ and $\{f_{xy,i,j}\}$, that is, $\{F(x,y)\}$, $\{\frac{\partial}{\partial x}F(x,y)\}$, $\{\frac{\partial}{\partial y}F(x,y)\}$ $\{\frac{\partial^2}{\partial x \partial y}F(x,y)\}$ at the nodes $\{(x_i, y_j)\}$, the piecewise cubic Hermite interpolant is $H(x,y) = H_{i,j}(x,y)$ if $(x,y) \in [x_i, x_{i+1}] \times [y_j, y_{j+1}]$, where:

$$
\begin{aligned}
H_{i,j}(x,y) \;=\; & f_{i,j}h_{1,i}(x)h_{1,j}(y) + f_{i+1,j}h_{2,i}(x)h_{1,j}(y) \\
& + f_{i,j+1}h_{1,i}(x)h_{2,j}(y) + f_{i+1,j+1}h_{2,i}(x)h_{2,j}(y) \\
& + f_{x,i,j}h_{3,i}(x)h_{1,j}(y) + f_{x,i+1,j}h_{4,i}(x)h_{1,j}(y) \\
& + f_{x,i,j+1}h_{3,i}(x)h_{2,j}(y) + f_{x,i+1,j+1}h_{4,i}(x)h_{2,j}(y) \\
& + f_{y,i,j}h_{1,i}(x)h_{3,j}(y) + f_{y,i+1,j}h_{2,i}(x)h_{3,j}(y) \\
& + f_{y,i,j+1}h_{1,i}(x)h_{4,j}(y) + f_{y,i+1,j+1}h_{2,i}(x)h_{4,j}(y) \\
& + f_{xy,i,j}h_{3,i}(x)h_{3,j}(y) + f_{xy,i+1,j}h_{4,i}(x)h_{3,j}(y) \\
& + f_{xy,i,j+1}h_{3,i}(x)h_{4,j}(y) + f_{xy,i+1,j+1}h_{4,i}(x)h_{4,j}(y)
\end{aligned}
$$

Piecewise Bicubic Hermite Interpolation

being

$$\begin{aligned}
h_{1,i}(x) &= 2(\tfrac{x-x_i}{x_{i+1}-x_i})^3 - 3(\tfrac{x-x_i}{x_{i+1}-x_i})^2 + 1 \\
h_{2,i}(x) &= -2(\tfrac{x-x_i}{x_{i+1}-x_i})^3 + 3(\tfrac{x-x_i}{x_{i+1}-x_i})^2 \\
h_{3,i}(x) &= ((\tfrac{x-x_i}{x_{i+1}-x_i})^3 - 2(\tfrac{x-x_i}{x_{i+1}-x_i})^2 + (\tfrac{x-x_i}{x_{i+1}-x_i}))(x_{i+1} - x_i) \\
h_{4,i}(x) &= ((\tfrac{x-x_i}{x_{i+1}-x_i})^3 - (\tfrac{x-x_i}{x_{i+1}-x_i})^2)(x_{i+1} - x_i)
\end{aligned}$$

and

$$\begin{aligned}
h_{1,j}(y) &= 2(\tfrac{y-y_j}{y_{j+1}-y_j})^3 - 3(\tfrac{y-y_j}{y_{j+1}-y_j})^2 + 1 \\
h_{2,j}(y) &= -2(\tfrac{y-y_j}{y_{j+1}-y_j})^3 + 3(\tfrac{y-y_j}{y_{j+1}-y_j})^2 \\
h_{3,j}(y) &= ((\tfrac{y-y_j}{y_{j+1}-y_j})^3 - 2(\tfrac{y-y_j}{y_{j+1}-y_j})^2 + (\tfrac{y-y_j}{y_{j+1}-y_j}))(y_{j+1} - y_j) \\
h_{4,j}(y) &= ((\tfrac{y-y_j}{y_{j+1}-y_j})^3 - (\tfrac{y-y_j}{y_{j+1}-y_j})^2)(y_{j+1} - y_j).
\end{aligned}$$

Piecewise Bicubic Hermite Interpolation. Accuracy

Accuracy with exact values PBH

If $F(x,y)$ is smooth and $f_{i,j} = F(x_i, y_j)$, $f_{x,i,j} = \frac{\partial}{\partial x} F(x_i, y_j)$, $f_{y,i,j} = \frac{\partial}{\partial y} F(x_i, y_j)$ and $f_{xy,i,j} = \frac{\partial^2}{\partial x \partial y} F(x_i, y_j)$ then

$$H(x,y) = F(x,y) + \mathcal{O}(h^4)$$

Accuracy with approximate values PBH

If $F(x,y)$ is smooth and $f_{i,j} = F(x_i, y_j) + \mathcal{O}(h^{p_1})$, $f_{x,i,j} = \frac{\partial}{\partial x} F(x_i, y_j) + \mathcal{O}(h^{p_2})$, $f_{y,i,j} = \frac{\partial}{\partial y} F(x_i, y_j) + \mathcal{O}(h^{p_3})$ and $f_{xy,i,j} = \frac{\partial^2}{\partial x \partial y} F(x_i, y_j) + \mathcal{O}(h^{p_4})$ then

$$H(x,y) = F(x,y) + \mathcal{O}(h^{\min\{4, p_1, p_2-1, p_3-1, p_4-2\}})$$

Piecewise Bicubic Hermite Interpolation

Data

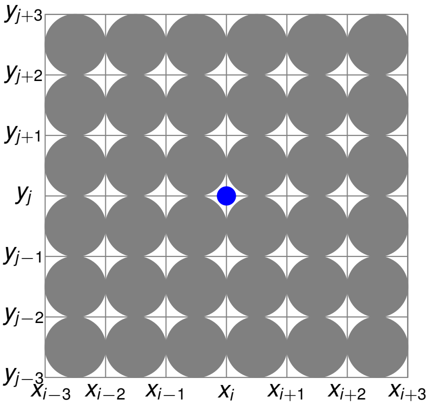
Here we assume that we know

$$m_{i,j} = \frac{1}{h^2} \int_{x_{i-1}}^{x_i} \int_{y_{j-1}}^{y_j} F(x,y)dxdy$$

Goal

A way to compute $\{f_{i,j}\}$, $\{f_{x,i,j}\}$, $\{f_{y,i,j}\}$ and $\{f_{xy,i,j}\}$ defines an algorithm for constructing a cubic Hermite interpolant.

Linear techniques.



From $m_{k,\ell}$, $k = i-2 : i+3$, $\ell = j-2 : j+3$, where

$m_{k,\ell} = \int_{x_{k-1}}^{x_k} \int_{y_{\ell-1}}^{y_\ell} F(x,y)\, dxdy$,

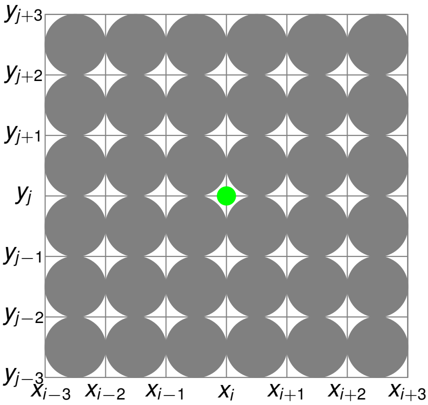
we construct $p^5_{i-2,j-2}$ such that

$$\int_{x_{k-1}}^{x_k} \int_{y_{\ell-1}}^{y_\ell} p^5_{i-2,j-2}(x,y)dxdy = m_{k,\ell}$$

and define

$$f^{5,5}_{i,j} = p^5_{i-2,j-2}(x_i, y_j)$$
$$f^{5,5}_{x,i,j} = \frac{\partial}{\partial x}\, p^5_{i-2,j-2}(x_i, y_j)$$
$$f^{5,5}_{y,i,j} = \frac{\partial}{\partial y}\, p^5_{i-2,j-2}(x_i, y_j)$$
$$f^{5,5}_{xy,i,j} = \frac{\partial^2}{\partial x \partial y}\, p^5_{i-2,j-2}(x_i, y_j)$$

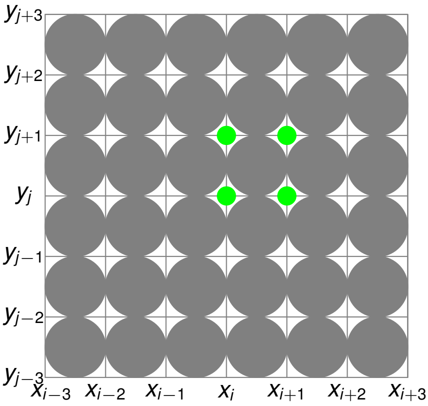
Linear techniques. If $F(x,y)$ is smooth in $[x_{i-3},x_{i+3}]\times y_{j-3},y_{j+3}]$



Accuracy

$$f^{5,5}_{i,j} = F(x_i,y_j) + \mathcal{O}(h^6);$$
$$f^{5,5}_{x,i,j} = \frac{\partial}{\partial x}F(x_i,y_j) + \mathcal{O}(h^5);$$
$$f^{5,5}_{y,i,j} = \frac{\partial}{\partial y}F(x_i,y_j) + \mathcal{O}(h^5)\ ;$$
$$f^{5,5}_{xy,i,j} = \frac{\partial^2}{\partial x\partial y}F(x_i,y_j) + \mathcal{O}(h^4)$$

Linear techniques. If $F(x,y)$ is smooth in $[x_{i-3}, x_{i+4}] \times y_{j-3}, y_{j+4}]$



Accuracy

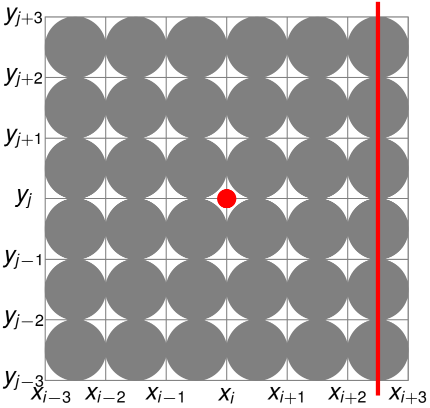
$$f^{5,5}_{k,\ell} = F(x_k, y_\ell) + \mathcal{O}(h^6);$$
$$f^{5,5}_{x,k,\ell} = \frac{\partial}{\partial x} F(x_k, y_\ell) + \mathcal{O}(h^5);$$
$$f^{5,5}_{y,k,\ell} = \frac{\partial}{\partial y} F(x_k, y_\ell) + \mathcal{O}(h^5)\ ;$$
$$f^{5,5}_{xy,k,\ell} = \frac{\partial^2}{\partial x \partial y} F(x_k, y_\ell) + \mathcal{O}(h^4),$$

for $k = i, i+1$, $\ell = j, j+1$. Then

$$H_{i,j}(x,y) = F(x,y) + \mathcal{O}(h^4)$$

in $[x_i, x_{i+1}] \times [y_j, y_{j+1}]$

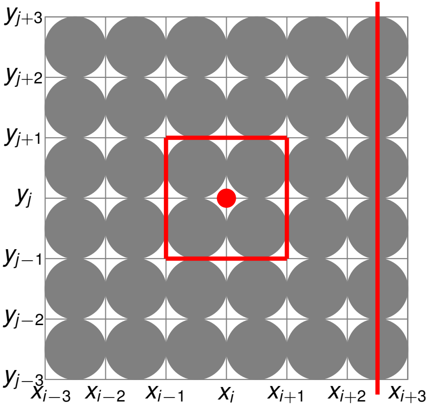
Linear techniques. Discontinuity



Accuracy

$$f_{i,j}^{5,5} = F(x_i, y_j) + \mathcal{O}(h^0);$$
$$f_{x,i,j}^{5,5} = \frac{\partial}{\partial x} F(x_i, y_j) + \mathcal{O}(h^{-1});$$
$$f_{y,i,j}^{5,5} = \frac{\partial}{\partial y} F(x_i, y_j) + \mathcal{O}(h^{-1}) \ ;$$
$$f_{xy,i,j}^{5,5} = \frac{\partial^2}{\partial x \partial y} F(x_i, y_j) + \mathcal{O}(h^{-2})$$

Linear techniques. Discontinuity



Accuracy

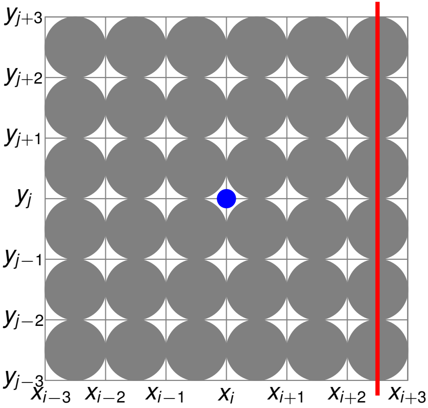
$$f^{5,5}_{i,j} = F(x_i, y_j) + \mathcal{O}(h^0);$$
$$f^{5,5}_{x,i,j} = \frac{\partial}{\partial x} F(x_i, y_j) + \mathcal{O}(h^{-1});$$
$$f^{5,5}_{y,i,j} = \frac{\partial}{\partial y} F(x_i, y_j) + \mathcal{O}(h^{-1}) \; ;$$
$$f^{5,5}_{xy,i,j} = \frac{\partial^2}{\partial x \partial y} F(x_i, y_j) + \mathcal{O}(h^{-2})$$

then

$$H(x, y) = F(x, y) + \mathcal{O}(h^0)$$

in $[x_{i-1}, x_{i+1}] \times [y_{j-1}, y_{j+1}]$.

Nonlinear techniques. Discontinuity



Goal

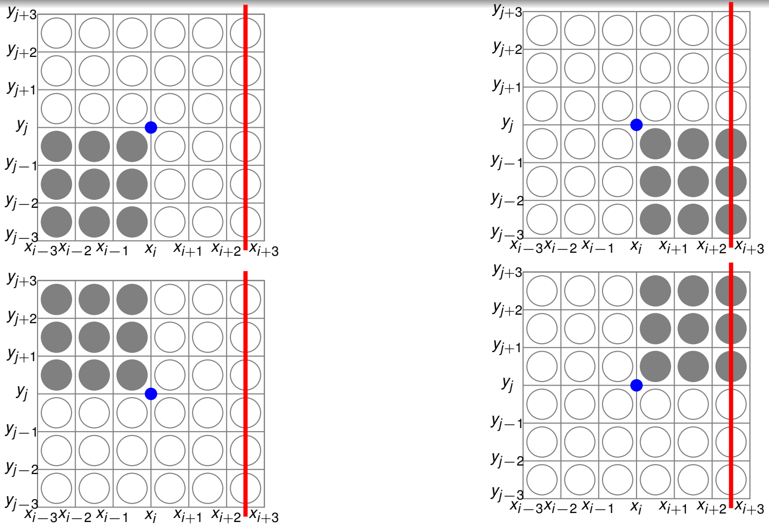
$f_{i,j} \approx F(x_i, y_j)$;
$f_{x,i,j} \approx \frac{\partial}{\partial x} F(x_i, y_j)$;
$f_{y,i,j} \approx \frac{\partial}{\partial y} F(x_i, y_j)$;
$f_{xy,i,j} \approx \frac{\partial^2}{\partial x \partial y} F(x_i, y_j)$

and

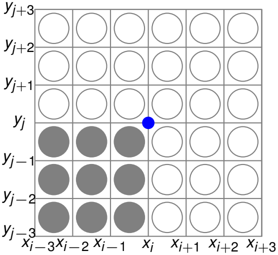
$$H(x, y) \approx F(x, y)$$

in the subintervals not affected by the discontinuity

Nonlinear techniques

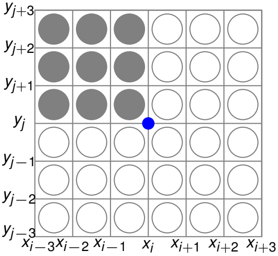


Nonlinear techniques



From $m_{k,\ell}$, $k = i-2 : i$, $\ell = j-2 : j$, we construct $p^2_{i-2,j-2}$ such that $\int_{x_{k-1}}^{x_k} \int_{y_{\ell-1}}^{y_\ell} p^2_{i-2,j-2}(x,y)dxdy = m_{k,\ell}$ and define

$$
\begin{aligned}
f^{0,0}_{i,j} &= p^2_{i-2,j-2}(x_i, y_j) \\
f^{0,0}_{x,i,j} &= \frac{\partial}{\partial x}\, p^2_{i-2,j-2}(x_i, y_j) \\
f^{0,0}_{y,i,j} &= \frac{\partial}{\partial y}\, p^2_{i-3,j-3}(x_i, y_j) \\
f^{0,0}_{xy,i,j} &= \frac{\partial^2}{\partial x \partial y}\, p^2_{i-3,j-3}(x_i, y_j)
\end{aligned}
$$



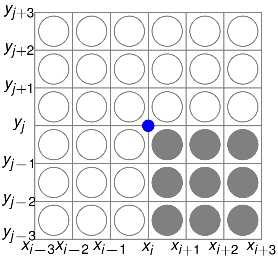
From $m_{k,\ell}$, $k = i-2 : i$, $\ell = j+1 : j+3$, we construct $p^2_{i-2,j+1}$ such that $\int_{x_{k-1}}^{x_k} \int_{y_{\ell-1}}^{y_\ell} p^5_{i-2,j+1}(x,y)dxdy = m_{k,\ell}$ and define

$$
\begin{aligned}
f^{0,3}_{i,j} &= p^2_{i-2,j+1}(x_i, y_j) \\
f^{0,3}_{x,i,j} &= \frac{\partial}{\partial x}\, p^2_{i-2,j+1}(x_i, y_j) \\
f^{0,3}_{y,i,j} &= \frac{\partial}{\partial y}\, p^2_{i-2,j+}(x_i, y_j) \\
f^{0,3}_{xy,i,j} &= \frac{\partial^2}{\partial x \partial y}\, p^2_{i-2,j+1}(x_i, y_j)
\end{aligned}
$$

Nonlinear techniques

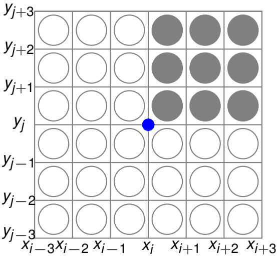
From $m_{k,\ell}$, $k = i+1 : i+3$, $\ell = j-2 : j$, we construct $p^2_{i+1,j-2}$ such that $\int_{x_{k-1}}^{x_k} \int_{y_{\ell-1}}^{y_\ell} p^2_{i+1,j-2}(x,y)dxdy = m_{k,\ell}$ and define

$$
\begin{aligned}
f^{3,0}_{i,j} &= p^2_{i+1,j-2}(x_i,y_j)\\
f^{3,0}_{x,i,j} &= \frac{\partial}{\partial x}\ p^2_{i+1,j-2}(x_i,y_j)\\
f^{3,0}_{y,i,j} &= \frac{\partial}{\partial y}\ p^2_{i+1,j-3}(x_i,y_j)\\
f^{3,0}_{xy,i,j} &= \frac{\partial^2}{\partial x \partial y}\ p^2_{i+1,j-3}(x_i,y_j)
\end{aligned}
$$



From $m_{k,\ell}$, $k = i+1 : i+3$, $\ell = j+1 : j+3$, we construct $p^5_{i+1,j+1}$ such that $\int_{x_{k-1}}^{x_k} \int_{y_{\ell-1}}^{y_\ell} p^2_{i+1,j+1}(x,y)dxdy = m_{k,\ell}$ and define

$$
\begin{aligned}
f^{3,3}_{i,j} &= p^2_{i+1,j+1}(x_i,y_j)\\
f^{3,3}_{x,i,j} &= \frac{\partial}{\partial x}\ p^2_{i+1,j+1}(x_i,y_j)\\
f^{3,3}_{y,i,j} &= \frac{\partial}{\partial y}\ p^2_{i+1,j+}(x_i,y_j)\\
f^{3,3}_{xy,i,j} &= \frac{\partial^2}{\partial x \partial y}\ p^2_{i+1,j+1}(x_i,y_j)
\end{aligned}
$$



WENO techniques. Optimal weights. Central WENO

Now we define

$$
\begin{aligned}
\widehat{f}_{i,j} &= 2\left(f^{5,5}_{i,j} - \left(f^{0,0}_{i,j} + f^{0,3}_{i,j} + f^{3,0}_{i,j} + f^{3,3}_{i,j}\right)\frac{1}{8}\right)\\
\widehat{f}_{x,i,j} &= 2\left(f^{5,5}_{x,i,j} - \left(f^{0,0}_{x,i,j} + f^{0,3}_{x,i,j} + f^{3,0}_{x,i,j} + f^{3,3}_{x,i,j}\right)\frac{1}{8}\right)\\
\widehat{f}_{y,i,j} &= 2\left(f^{5,5}_{y,i,j} - \left(f^{0,0}_{y,i,j} + f^{0,3}_{y,i,j} + f^{3,0}_{y,i,j} + f^{3,3}_{y,i,j}\right)\frac{1}{8}\right)\\
\widehat{f}_{xy,i,j} &= 2\left(f^{5,5}_{xy,i,j} - \left(f^{0,0}_{xy,i,j} + f^{0,3}_{xy,i,j} + f^{3,0}_{xy,i,j} + f^{3,3}_{xy,i,j}\right)\frac{1}{8}\right)\\
f^{5,5}_{i,j} &= \frac{1}{2}\widehat{f}_{i,j} + \frac{1}{8}f^{0,0}_{i,j} + \frac{1}{8}f^{0,3}_{i,j} + \frac{1}{8}f^{3,0}_{i,j} + \frac{1}{8}f^{3,3}_{i,j}\\
f^{5,5}_{x,i,j} &= \frac{1}{2}\widehat{f}_{x,i,j} + \frac{1}{8}f^{0,0}_{x,i,j} + \frac{1}{8}f^{0,3}_{x,i,j} + \frac{1}{8}f^{3,0}_{x,i,j} + \frac{1}{8}f^{3,3}_{x,i,j}\\
f^{5,5}_{y,i,j} &= \frac{1}{2}\widehat{f}_{y,i,j} + \frac{1}{8}f^{0,0}_{y,i,j} + \frac{1}{8}f^{0,3}_{y,i,j} + \frac{1}{8}f^{3,0}_{y,i,j} + \frac{1}{8}f^{3,3}_{y,i,j}\\
f^{5,5}_{xy,i,j} &= \frac{1}{2}\widehat{f}_{xy,i,j} + \frac{1}{8}f^{0,0}_{xy,i,j} + \frac{1}{8}f^{0,3}_{xy,i,j} + \frac{1}{8}f^{3,0}_{xy,i,j} + \frac{1}{8}f^{3,3}_{xy,i,j}
\end{aligned}
$$

WENO techniques. Nonlinear Weights

We have

$$f^{5,5}_{i,j} = \frac{1}{2}\widehat{f}_{i,j} + \frac{1}{8} f^{0,0}_{i,j} + \frac{1}{8} f^{0,3}_{i,j} + \frac{1}{8} f^{3,0}_{i,j} + \frac{1}{8} f^{3,3}_{i,j}$$

We define $\widehat{\omega}_{i,j}$, $\omega^{k,\ell}_{i,j}$, $k,\ell = 0,3$ and

$$f^{\omega}_{i,j} = \widehat{\omega}_{i,j}\ \widehat{f}_{i,j} + \omega^{0,0}_{i,j}\ f^{0,0}_{i,j} + \omega^{0,3}_{i,j}\ f^{0,3}_{i,j} + \omega^{3,0}_{i,j}\ f^{3,0}_{i,j} + \omega^{3,3}_{i,j}\ f^{3,3}_{i,j}$$

Goal: If $F(x,y)$ is smooth in $[x_{i-3}, x_{i+3}] \times [y_{j-3}, y_{j+3}]$

$\widehat{\omega}_{i,j} \approx \frac{1}{2}$, $\omega^{k,\ell}_{i,j} \approx \frac{1}{8}$, $k,\ell = 0,3$ and $f^{w}_{i,j} \approx f^{5,5}_{i,j}$.

Goal: If $F(x,y)$ is smooth only in $[x_{i-3}, x_{i}] \times [y_{j-3}, y_{j}]$

$\omega^{0,0}_{i,j} \approx 1$, $\widehat{\omega}_{i,j} \approx 0$, $\omega^{0,3}_{i,j} \approx 0$, $\omega^{3,0}_{i,j} \approx 0$, $\omega^{3,3}_{i,j} \approx 0$, and $f^{w}_{i,j} \approx f^{0,0}_{i,j}$.

WENO techniques. Smoothness indicators

Shu et al. 1996-2021

$$IS_{i,j}^{k,\ell} = \sum_{l_1+l_2=1,2} h^{2(l_1+l_2-1)} \int_{x_i-h/2}^{x_i+h/2} \int_{y_j-h/2}^{y_j+h/2} \left(\frac{\partial^{l_1+l_2}}{\partial^{l_1} \partial^{l_2}} p^2_{i-2+k,j-2+\ell}(x,y) \right)^2 dxdy,$$

Aràndiga et al. 2010

$$IS_{i,j}^{k,\ell} = \sum_{l_1+l_2=1,2} h^{2(l_1+l_2)} \left(\frac{\partial^{l_1+l_2}}{\partial^{l_1} \partial^{l_2}} p^2_{i-2+k,j-2+\ell}(x_i,y_j) \right)^2 dxdy,$$

$$\widehat{IS}_{i,j} = IS_{i,j}^{0,0} + IS_{i,j}^{0,3} + IS_{i,j}^{3,0} + IS_{i,j}^{3,3}$$

WENO techniques. Nonlinear weights

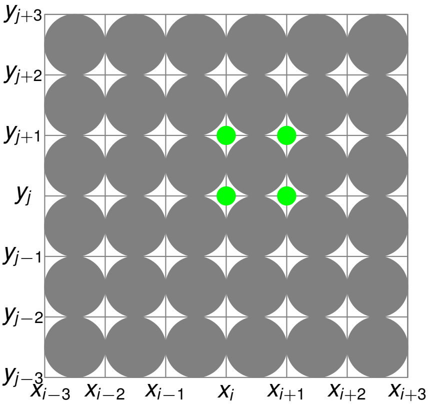
$$\alpha_{i,j}^{k,\ell} = \frac{1/8}{\left(h^2+IS_{i,j}^{k,\ell}\right)^2}, \; k,\ell = 0,3; \quad \widehat{\alpha}_{i,j} = \frac{1/2}{\left(h^2+\widehat{IS}_{i,j}\right)^2}.$$

$$\alpha_{i,j} = \widehat{\alpha}_{i,j} + \alpha_{i,j}^{0,0} + \alpha_{i,j}^{0,3} + \alpha_{i,j}^{3,0} + \alpha_{i,j}^{3,3}$$

$$\omega_{i,j}^{k,\ell} = \frac{\alpha_{i,j}^{k,\ell}}{\alpha_{i,j}}, \; k,\ell = 0,3; \quad \widehat{\omega}_{i,j} = \frac{\widehat{\alpha}_{i,j}}{\alpha_{i,j}}.$$

$$f_{i,j}^{w} = \widehat{\omega}_{i,j}\,\widehat{f}_{i,j} + \omega_{i,j}^{0,0}\, f_{i,j}^{0,0} + \omega_{i,j}^{0,3}\, f_{i,j}^{0,3} + \omega_{i,j}^{3,0}\, f_{i,j}^{3,0} + \omega_{i,j}^{3,3}\, f_{i,j}^{3,3}$$

If $F(x,y)$ is smooth in $[x_{i-3},x_{i+4}] \times y_{j-3},y_{j+4}]$



Accuracy

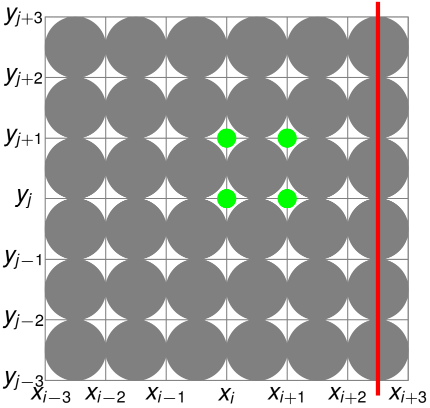
$$f^w_{k,\ell} = F(x_k,y_\ell) + \mathcal{O}(h^6);$$
$$f^w_{x,k,\ell} = \frac{\partial}{\partial x} F(x_k,y_\ell) + \mathcal{O}(h^5);$$
$$f^w_{y,k,\ell} = \frac{\partial}{\partial y} F(x_k,y_\ell) + \mathcal{O}(h^5)\ ;$$
$$f^w_{xy,k,\ell} = \frac{\partial^2}{\partial x \partial y} F(x_k,y_\ell) + \mathcal{O}(h^4),$$

for $k=i,i+1$, $\ell=j,j+1$. Then

$$H_{i,j}(x,y) = F(x,y) + \mathcal{O}(h^4)$$

in $[x_i,x_{i+1}] \times [y_j,y_{j+1}]$

If $F(x,y)$ has a discontinuity in $(x_{i+2}, x_{i+3}) \times [y_{j-3}, y_{j+3}]$.



Accuracy

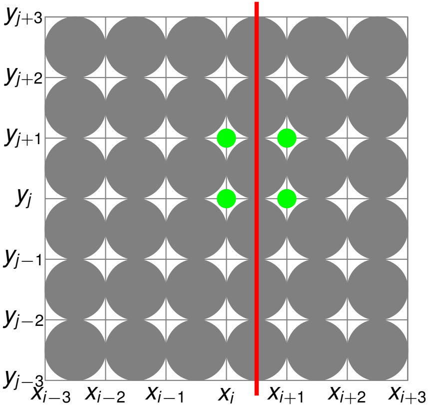
$$f^w_{k,\ell} = F(x_k, y_\ell) + \mathcal{O}(h^3);$$
$$f^w_{x.k,\ell} = \frac{\partial}{\partial x} F(x_k, y_\ell) + \mathcal{O}(h^2);$$
$$f^w_{y,k,\ell} = \frac{\partial}{\partial y} F(x_k, y_\ell) + \mathcal{O}(h^2)\ ;$$
$$f^w_{xy,k,\ell} = \frac{\partial^2}{\partial x \partial y} F(x_k, y_\ell) + \mathcal{O}(h^1),$$

for $k = i, i+1$, $\ell = j, j+1$. Then

$$H_{i,j}(x,y) = F(x,y) + \mathcal{O}(h^3)$$

in $[x_i, x_{i+1}] \times [y_j, y_{j+1}]$

If $F(x,y)$ has a discontinuity in $(x_i, x_{i+1}) \times [y_{j-3}, y_{j+3}]$.



Accuracy

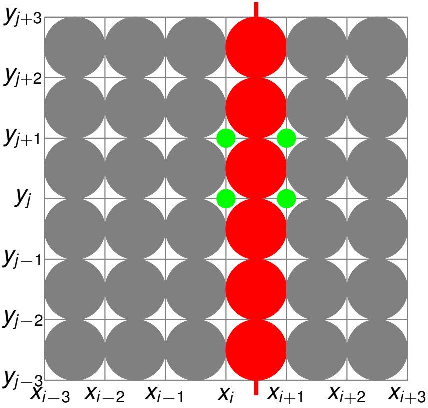
$$f^w_{k,\ell} = F(x_k, y_\ell) + \mathcal{O}(h^3);$$
$$f^w_{x,k,\ell} = \frac{\partial}{\partial x} F(x_k, y_\ell) + \mathcal{O}(h^2);$$
$$f^w_{y,k,\ell} = \frac{\partial}{\partial y} F(x_k, y_\ell) + \mathcal{O}(h^2)\ ;$$
$$f^w_{xy,k,\ell} = \frac{\partial^2}{\partial x \partial y} F(x_k, y_\ell) + \mathcal{O}(h^1),$$

for $k = i, i+1$, $\ell = j, j+1$. But

$$H_{i,j}(x,y) = F(x,y) + \mathcal{O}(h^0)$$

in $[x_i, x_{i+1}] \times [y_j, y_{j+1}]$

If $F(x, y)$ has a discontinuity in $(x_i, x_{i+1}) \times [y_{j-3}, y_{j+3}]$.



Accuracy

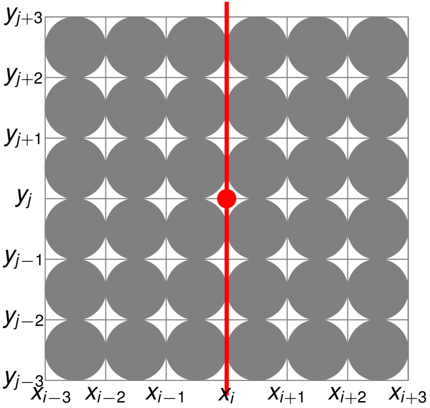
$$f^w_{k,\ell} = F(x_k, y_\ell) + \mathcal{O}(h^3);$$
$$f^w_{x,k,\ell} = \frac{\partial}{\partial x} F(x_k, y_\ell) + \mathcal{O}(h^2);$$
$$f^w_{y,k,\ell} = \frac{\partial}{\partial y} F(x_k, y_\ell) + \mathcal{O}(h^2) \;;$$
$$f^w_{xy,k,\ell} = \frac{\partial^2}{\partial x \partial y} F(x_k, y_\ell) + \mathcal{O}(h^1),$$

for $k = i, i+1$, $\ell = j, j+1$. But

$$H_{i,j}(x, y) = F(x, y) + \mathcal{O}(h^0)$$

in $[x_i, x_{i+1}] \times [y_j, y_{j+1}]$

If $F(x,y)$ has a discontinuity in $x_i \times [y_{j-3}, y_{j+3}]$.

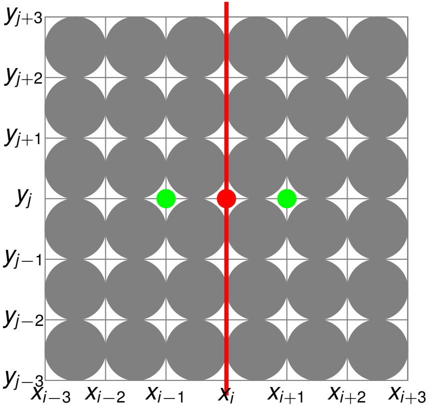


Accuracy

$$f^w_{k,\ell} = F(x_k, y_\ell) + \mathcal{O}(h^0);$$
$$f^w_{x,k,\ell} = \frac{\partial}{\partial x} F(x_k, y_\ell) + \mathcal{O}(h^{-1});$$
$$f^w_{y,k,\ell} = \frac{\partial}{\partial y} F(x_k, y_\ell) + \mathcal{O}(h^{-1}) \;;$$
$$f^w_{xy,k,\ell} = \frac{\partial^2}{\partial x \partial y} F(x_k, y_\ell) + \mathcal{O}(h^{-2}),$$

for $k = i$, $\ell = j$.

If $F(x, y)$ has a discontinuity in $x_i \times [y_{j-3}, y_{j+3}]$.

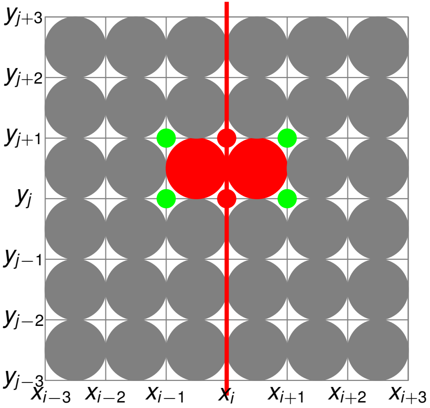


Accuracy

$$f^w_{k,\ell} = F(x_k, y_\ell) + \mathcal{O}(h^3);$$
$$f^w_{x,k,\ell} = \frac{\partial}{\partial x} F(x_k, y_\ell) + \mathcal{O}(h^2);$$
$$f^w_{y,k,\ell} = \frac{\partial}{\partial y} F(x_k, y_\ell) + \mathcal{O}(h^2)\ ;$$
$$f^w_{xy,k,\ell} = \frac{\partial^2}{\partial x \partial y} F(x_k, y_\ell) + \mathcal{O}(h^1),$$

for $k = i-1, i+1$, $\ell = j$.

If $F(x,y)$ has a discontinuity in $x_i \times [y_{j-3}, y_{j+3}]$.



Accuracy

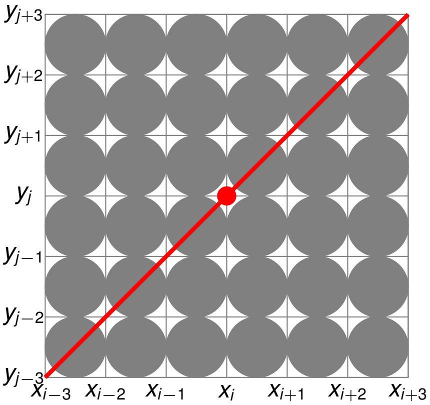
$$f^w_{k,\ell} = F(x_k, y_\ell) + \mathcal{O}(h^0);$$
$$f^w_{x,k,\ell} = \frac{\partial}{\partial x} F(x_k, y_\ell) + \mathcal{O}(h^{-1});$$
$$f^w_{y,k,\ell} = \frac{\partial}{\partial y} F(x_k, y_\ell) + \mathcal{O}(h^{-1}) \; ;$$
$$f^w_{xy,k,\ell} = \frac{\partial^2}{\partial x \partial y} F(x_k, y_\ell) + \mathcal{O}(h^{-2}),$$

for $k = i$, $\ell = j-3, j+3$. Then

$$H_{i,j}(x,y) = F(x,y) + \mathcal{O}(h^0)$$

in $[x_{i-1}, x_{i+1}] \times [y_{j-3}, y_{j+3}]$

If $F(x,y)$ has a discontinuity in $y=x$.

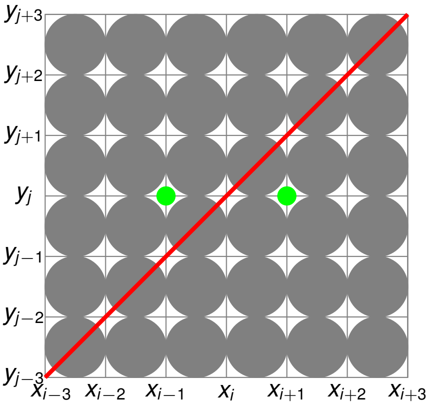


Accuracy

$$f^w_{k,\ell} = F(x_k, y_\ell) + \mathcal{O}(h^0);$$
$$f^w_{x,k,\ell} = \frac{\partial}{\partial x} F(x_k, y_\ell) + \mathcal{O}(h^{-1});$$
$$f^w_{y,k,\ell} = \frac{\partial}{\partial y} F(x_k, y_\ell) + \mathcal{O}(h^{-1})\ ;$$
$$f^w_{xy,k,\ell} = \frac{\partial^2}{\partial x \partial y} F(x_k, y_\ell) + \mathcal{O}(h^{-2}),$$

for $k = i$, $\ell = j$.

If $F(x, y)$ has a discontinuity in $y = x$.

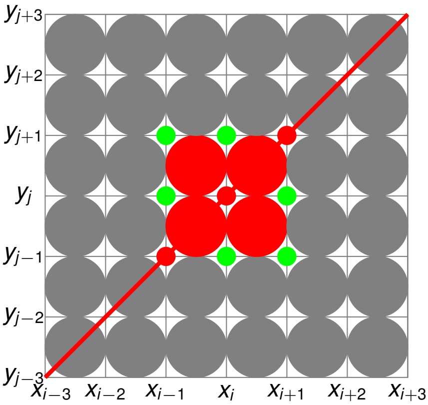


Accuracy

$$f^w_{k,\ell} = F(x_k, y_\ell) + \mathcal{O}(h^3);$$
$$f^w_{x,k,\ell} = \frac{\partial}{\partial x} F(x_k, y_\ell) + \mathcal{O}(h^2);$$
$$f^w_{y,k,\ell} = \frac{\partial}{\partial y} F(x_k, y_\ell) + \mathcal{O}(h^2)\ ;$$
$$f^w_{xy,k,\ell} = \frac{\partial^2}{\partial x \partial y} F(x_k, y_\ell) + \mathcal{O}(h^1),$$

for $k = i - 1, i + 1, \ell = j$.

If $F(x, y)$ has a discontinuity in $y = x$.



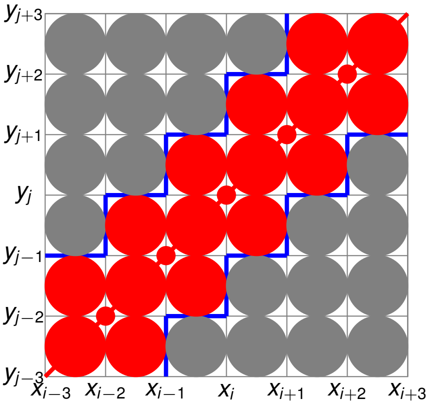
Accuracy

Then

$$H_{i,j}(x, y) = F(x, y) + \mathcal{O}(h^0)$$

in $[x_{i-1}, x_{i+1}] \times [y_{j-1}, y_{j+1}]$.

If $F(x,y)$ has a discontinuity in $y=x$.



Accuracy

Then

- $H_{i,j}(x,y) = F(x,y) + \mathcal{O}(h^0)$ between blue lines
- $H_{i,j}(x,y) = F(x,y) + \mathcal{O}(h^3)$ close to discontinuity
- $H_{i,j}(x,y) = F(x,y) + \mathcal{O}(h^4)$ away to discontinuity

Example 1.

Given the averages $m_{i,j} = \frac{1}{h^2}\int_{x_{i-1}}^{x_i}\int_{y_{j-1}}^{y_j} F(x,y)dxdy$ on a rectangular mesh $x_0 = 0$, $x_i = i\ h$, $y_0 = 0$, $y_j = j\ h$, $h = 1/16$, $i, j = 1, \ldots, 16$, where

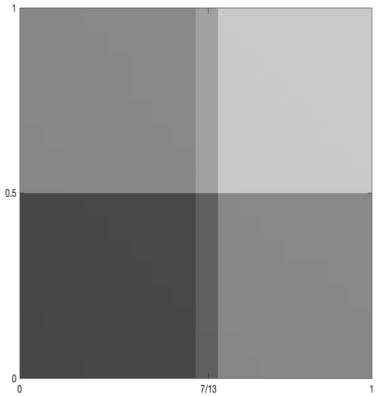
$$F(x,y) = \begin{cases} \sin(\pi(x+2)/8)\cos(\pi(y+2)/8), & 0 \le x \le .5; 0 \le y \le .5 \\ \sin(\pi(x+2)/8)\cos(\pi(y+2)/8), & .5 < x \le 1; 0 \le y \le .5 \\ \sin(\pi(x+2))/8\cos(\pi(y+2)/8), & 0 \le x \le .5; .5 < y \le 1 \\ \sin(\pi(x+2)/8)\cos(\pi(y+2)/8), & .5 < x \le 1; .5 < y \le 1 \end{cases}$$

we obtain an approximation $H(x,y)$ to $F(x,y)$ and compare

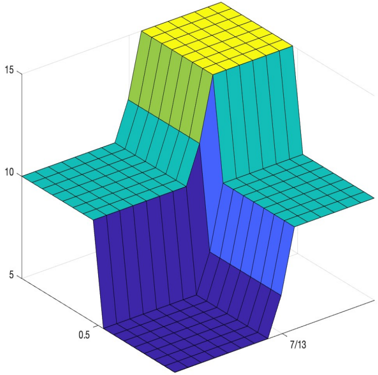
$$v_{i,j} = \frac{1}{h^2}\int_{x_{i-1}}^{x_i}\int_{y_{j-1}}^{y_j} H(x,y)dxdy$$

with $m_{i,j}$

Example 1. Linear techniques

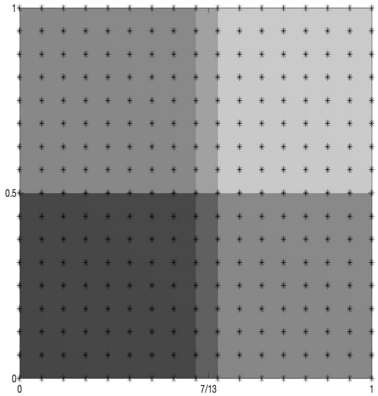


Original image

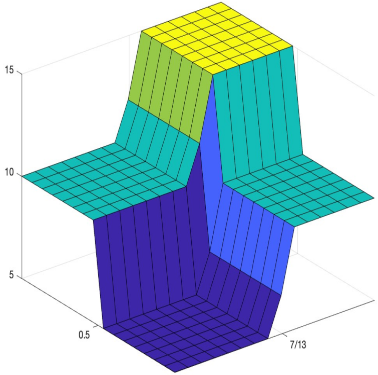


Original image

Example 1. Linear techniques

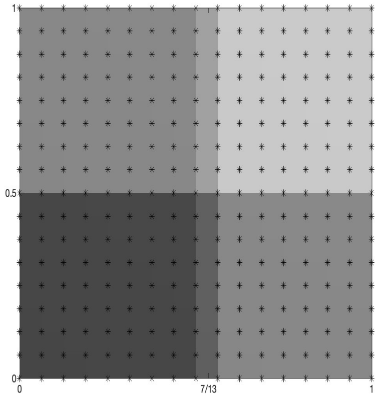


Original image.



Original image

Example 1. Linear techniques



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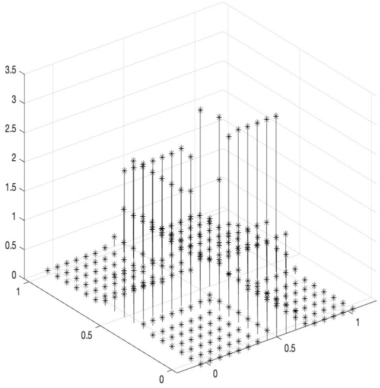
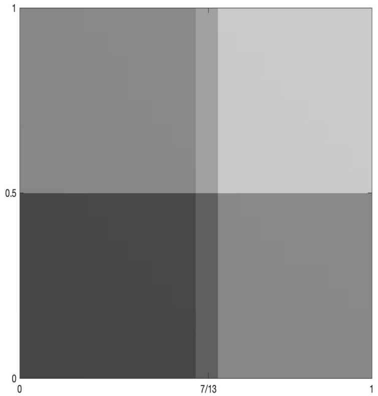


Error poinvalued obtained

Example 1. Linear techniques



Original image

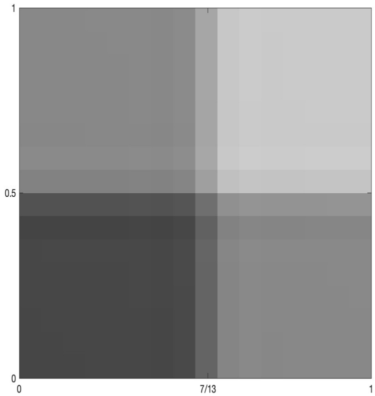
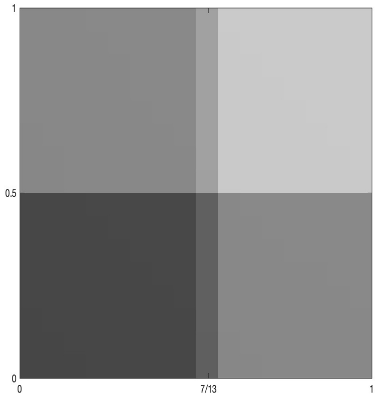
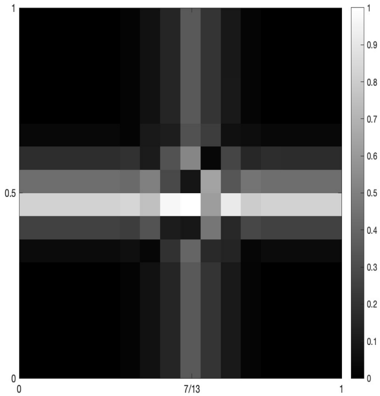


Image reconstructed

Example 1. Linear techniques

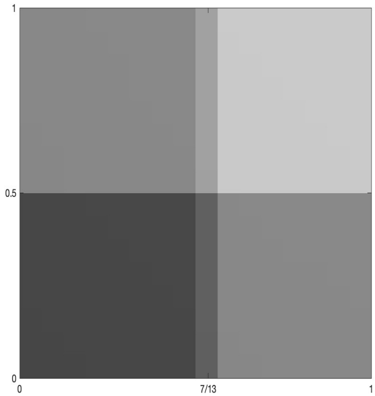


Original image.



Error original-reconstructed

Example 1. Linear techniques



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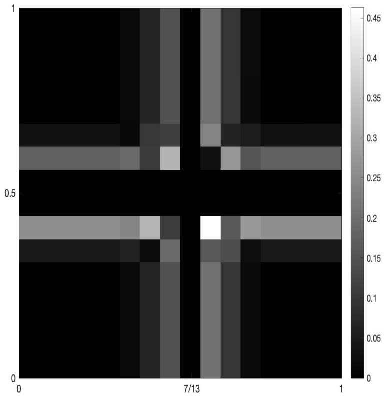
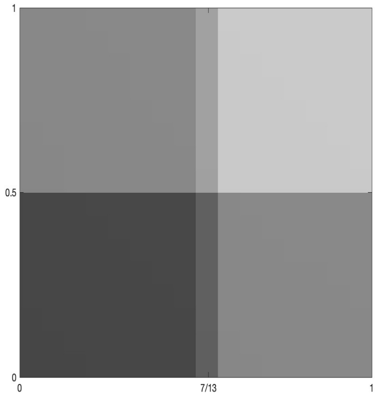
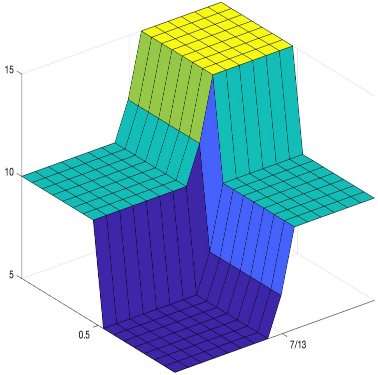


Error in non affected by disc

Example 1. Nonlinear techniques

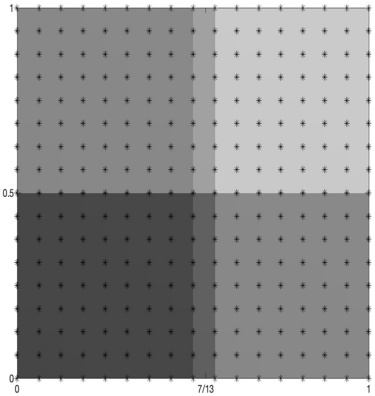


Original image



Original image

Example 1. Nonlinear techniques



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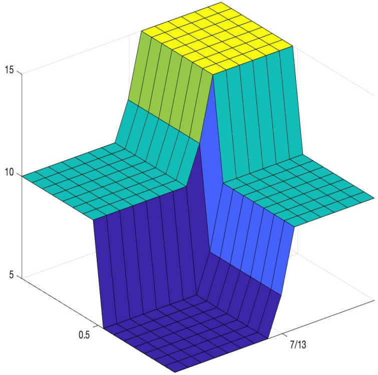
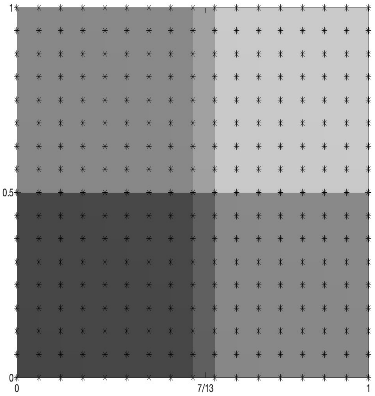


Original image

Example 1. Nonlinear techniques



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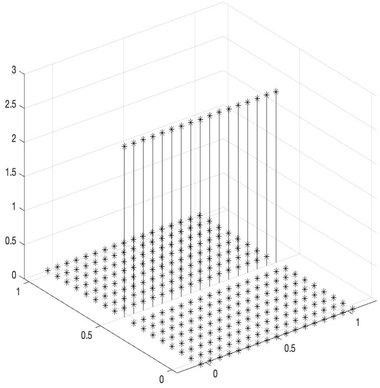
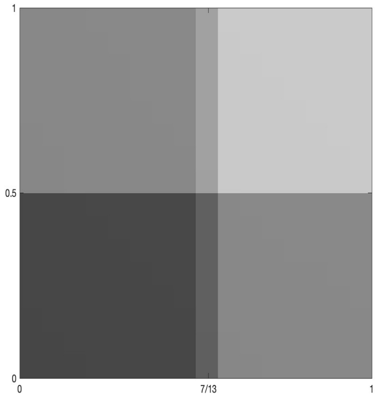


Error poinvalued obtained

Example 1. Nonlinear techniques



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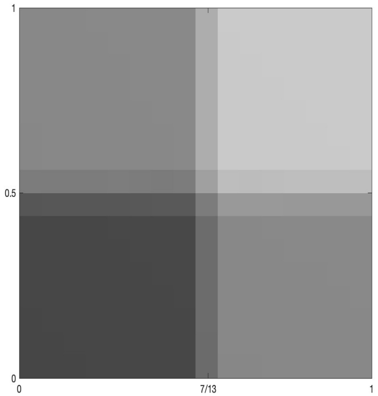
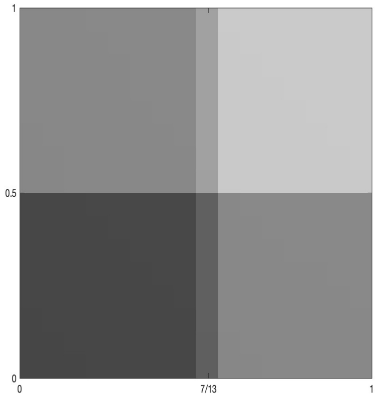
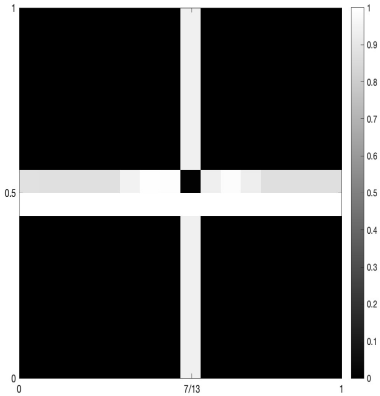


Image reconstructed

Example 1. Nonlinear techniques

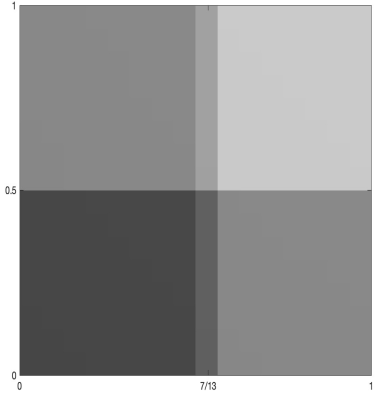


Original image.

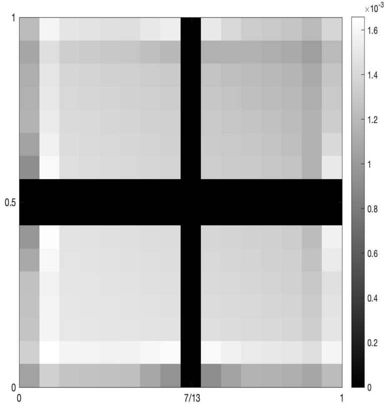


Error original-reconstructed

Example 1. Nonlinear techniques



Original image



Error in non affected by disc

Example 2

Given the averages $m_{i,j} = \frac{1}{h^2}\int_{x_{i-1}}^{x_i}\int_{y_{j-1}}^{y_j} F(x,y)dxdy$ on a rectangular mesh $x_0 = 0$, $x_i = i\ h$, $y_0 = 0$, $y_j = j\ h$, $h = 1/16$, $i, j = 1, \ldots, 16$, where

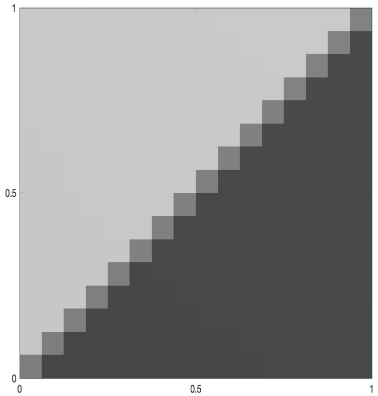
$$F(x,y) = \begin{cases} \sin(\pi(x+2)/8)\sin(\pi(y+2)/8) + 15, & 0 \leq x < y \leq 1, \\ \sin(\pi(x+2)/8)\sin(\pi(y+2)/8) + 5, & 0 \leq y \leq x \leq 1; \end{cases}$$

we obtain an approximation $H(x,y)$ to $F(x,y)$ and compare

$$v_{i,j} = \frac{1}{h^2}\int_{x_{i-1}}^{x_i}\int_{y_{j-1}}^{y_j} H(x,y)dxdy$$

with $m_{i,j}$

Example 2. Linear techniques



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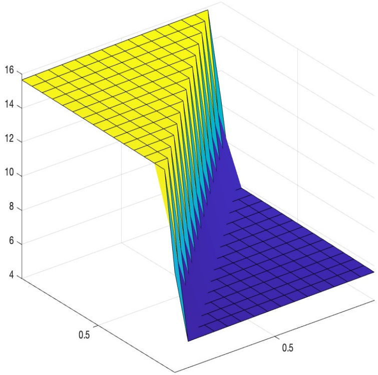
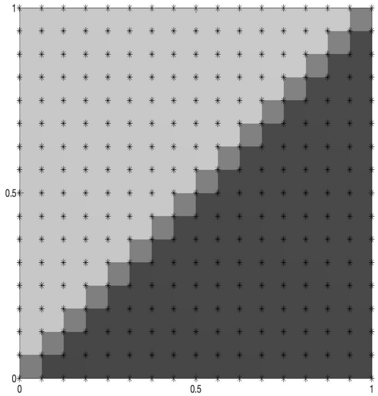


Original image

Example 2. Linear techniques



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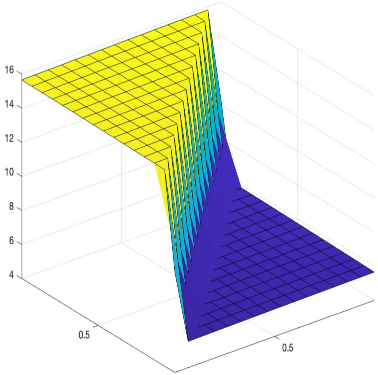

Original image

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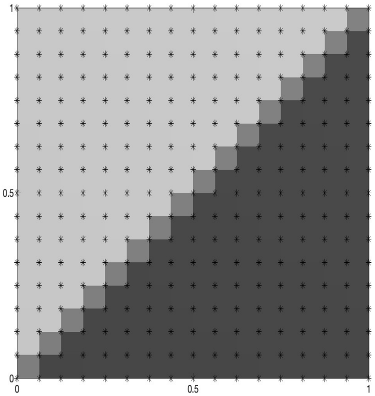

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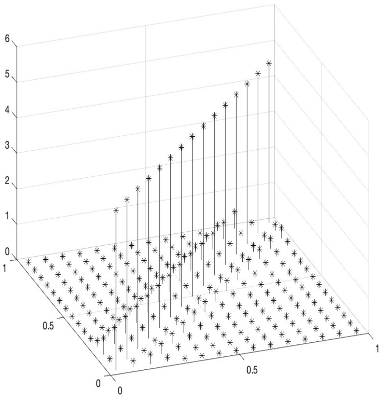
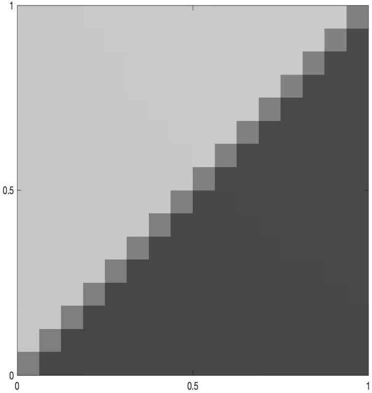


Error poinvalued obtained

Example 2. Linear techniques



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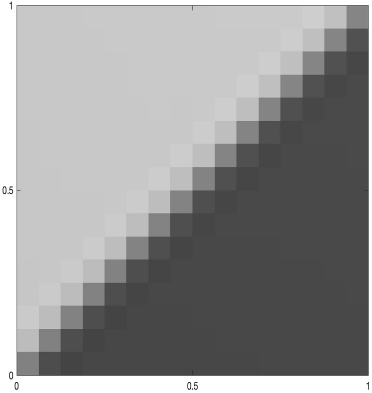
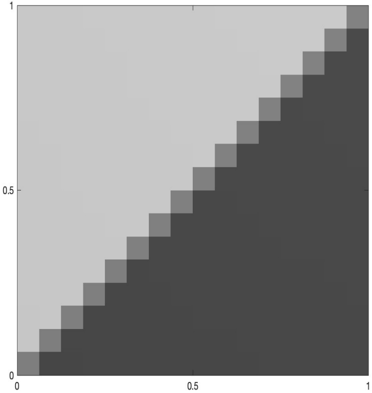


Image reconstructed

Example 2. Linear techniques



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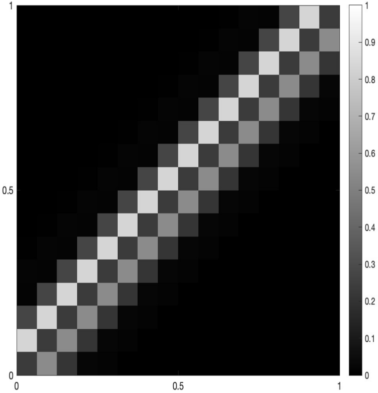
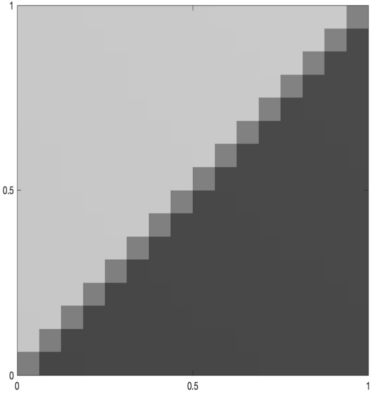


Error original-reconstructed

Example 2. Linear techniques



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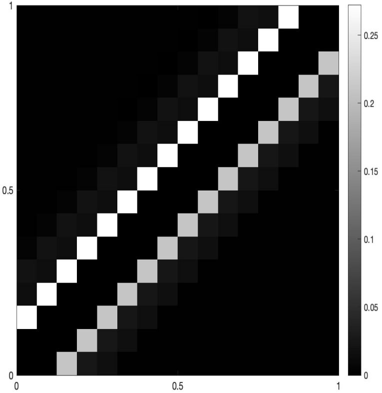
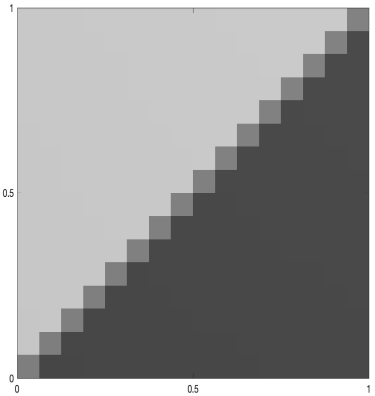


Error in non affected by disc

Example 2. Nonlinear techniques



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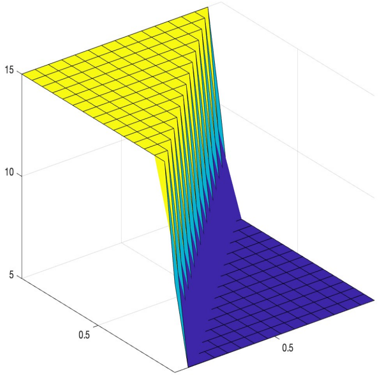

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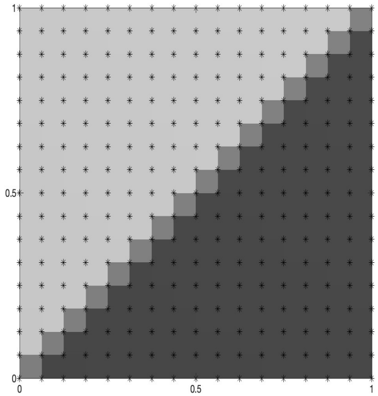

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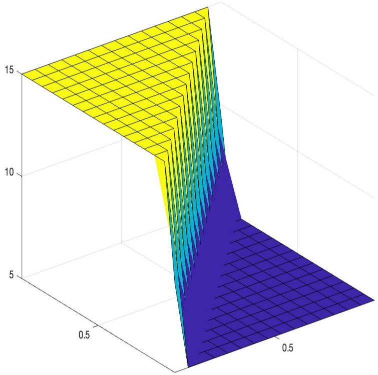

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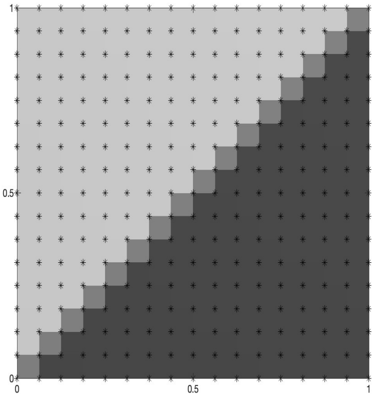
Example 2. Nonlinear techniques



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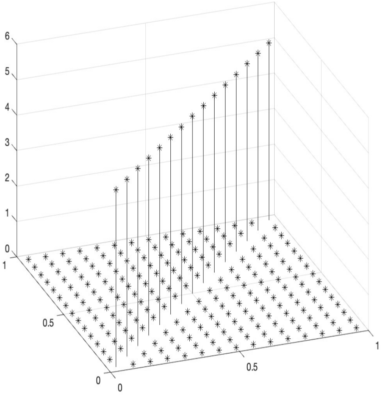
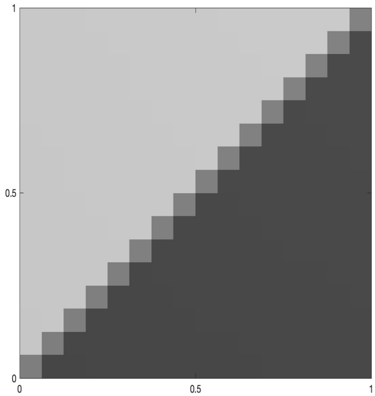


Error poinvalued obtained

Example 2. Nonlinear techniques



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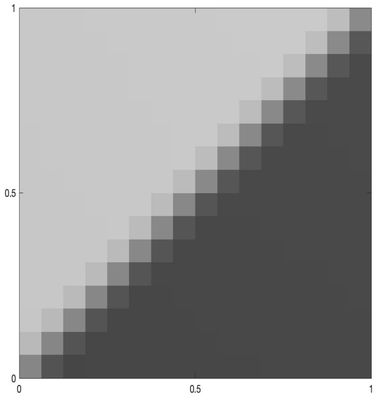
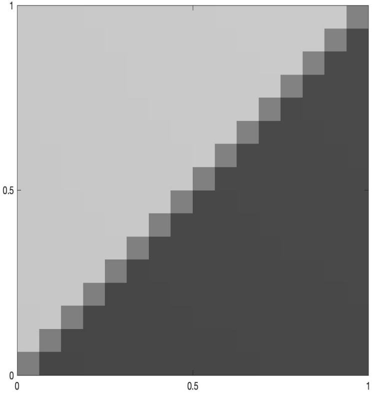


Image reconstructed

Example 2. Nonlinear techniques



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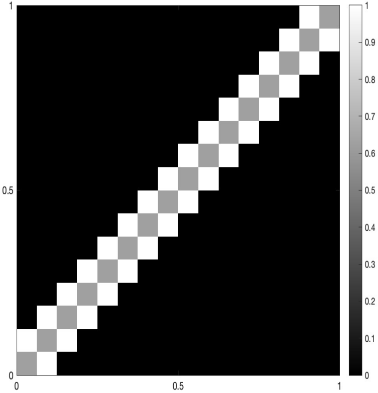
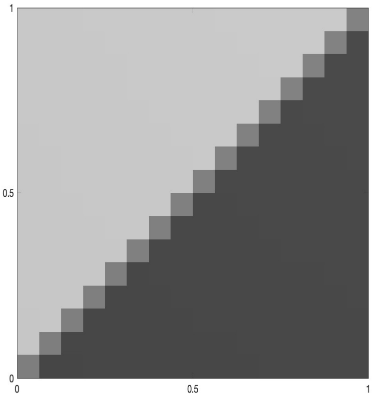


Error original-reconstructed

Example 2. Nonlinear techniques



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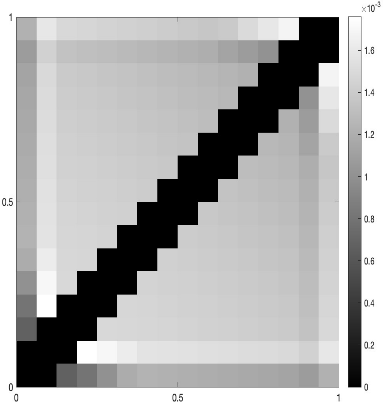


Error in non affected by disc

Example 3

Given the averages $m_{i,j} = \frac{1}{h^2}\int_{x_{i-1}}^{x_i}\int_{y_{j-1}}^{y_j} F(x,y)dxdy$ on a rectangular mesh $x_0 = 0$, $x_i = i\ h$, $y_0 = 0$, $y_j = j\ h$, $h = 1/16$, $i,j = 1,\dots,16$, where

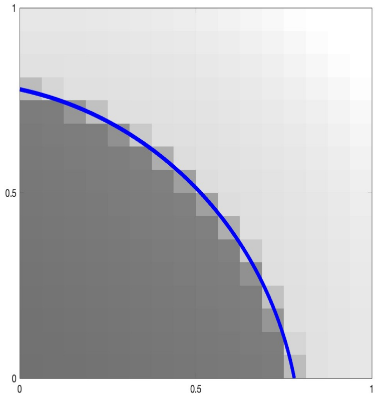
$$F(x,y) = \begin{cases} \exp(x+y)+10, & (x+.2)^2+(y+.2)^2 < 1, \\ \exp(x^2+y^2)+20, & (x+.2)^2+(y+.2)^2 \geq 1, \end{cases}$$

we obtain an approximation $H(x,y)$ to $F(x,y)$ and compare

$$v_{i,j} = \frac{1}{h^2}\int_{x_{i-1}}^{x_i}\int_{y_{j-1}}^{y_j} H(x,y)dxdy$$

with $m_{i,j}$

Example 3. Linear techniques



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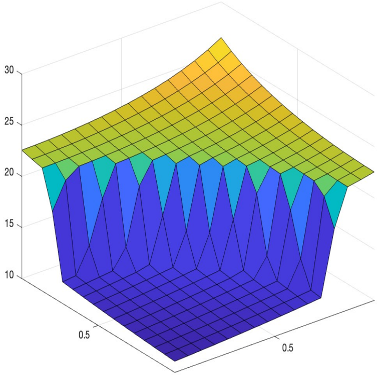

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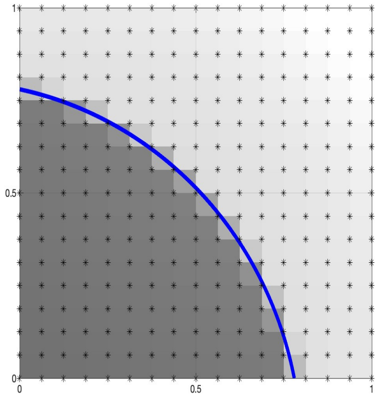
Example 3. Linear techniques



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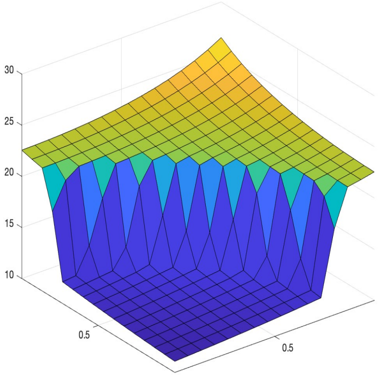

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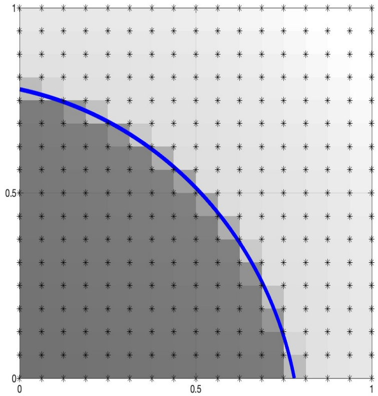

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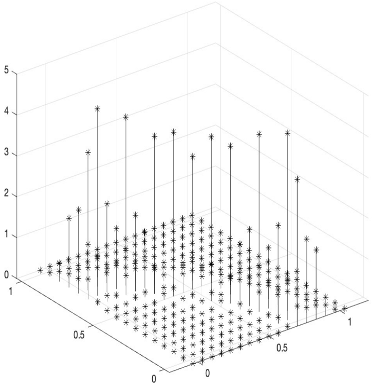

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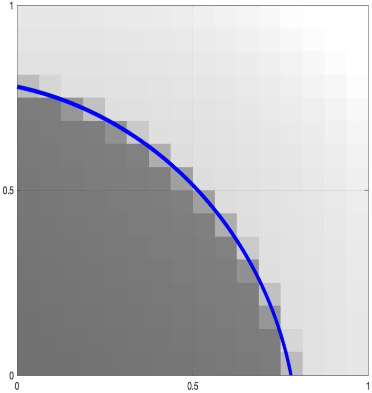
Example 3. Linear techniques



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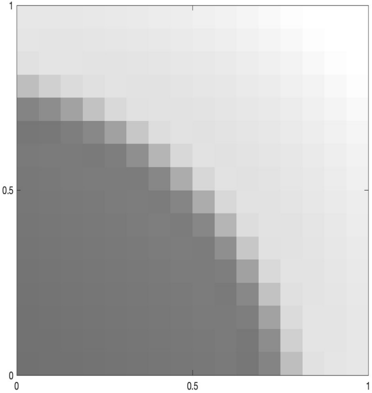
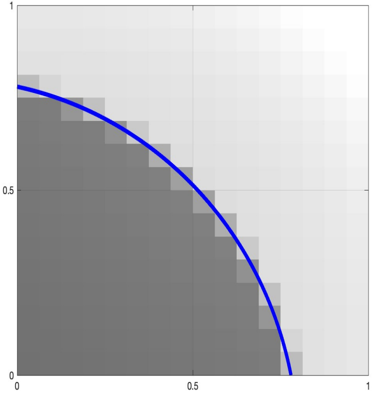


Image reconstructed

Example 3. Linear techniques



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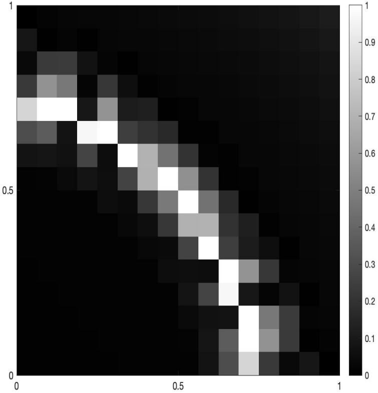
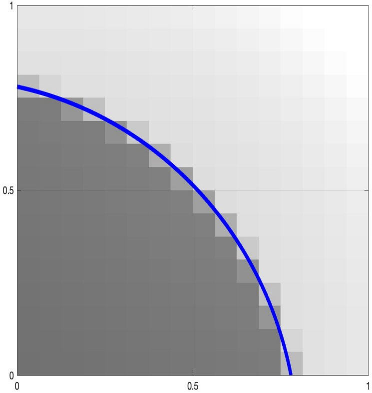


Error original-reconstructed

Example 3. Linear techniques



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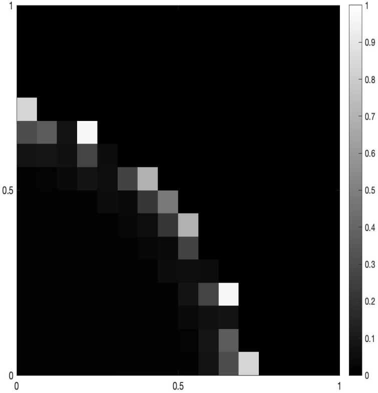
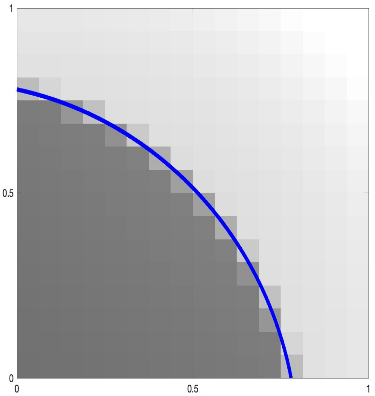


Error in non affected by disc

Example 3. Nonlinear techniques



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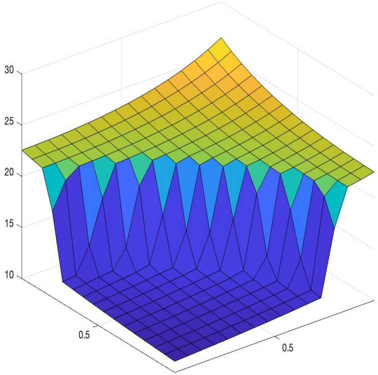
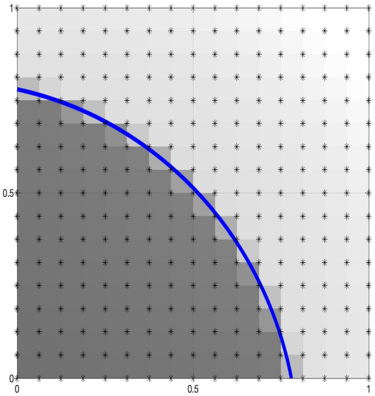


Original image

Example 3. Nonlinear techniques



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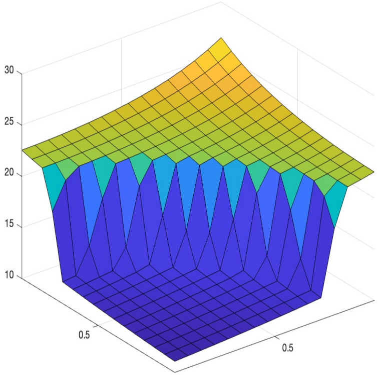

Original image

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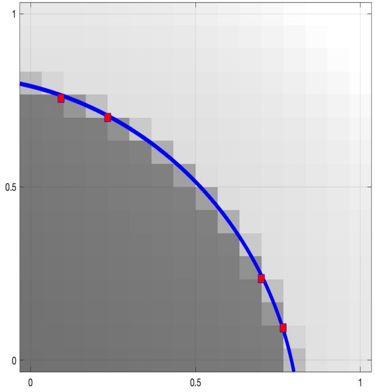

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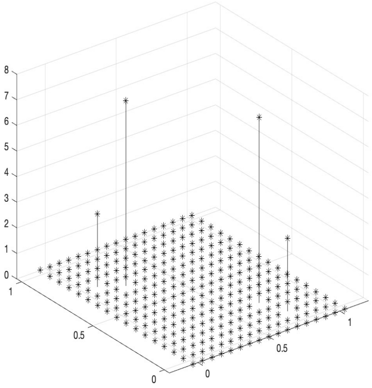

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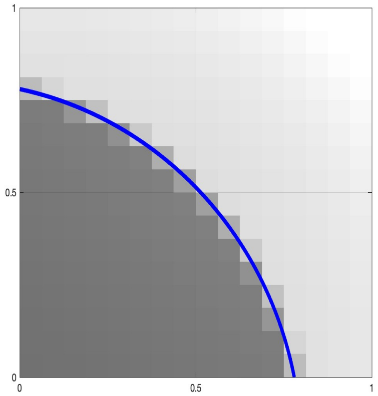
Example 3. Nonlinear techniques



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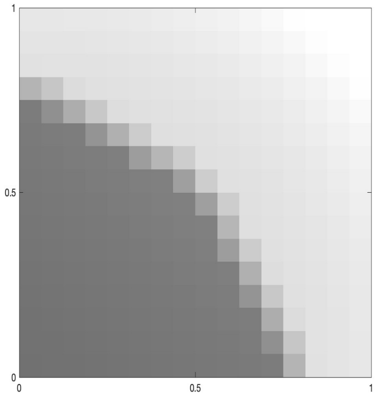
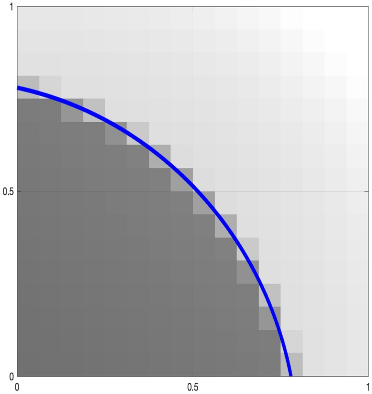


Image reconstructed

Example 3. Nonlinear techniques



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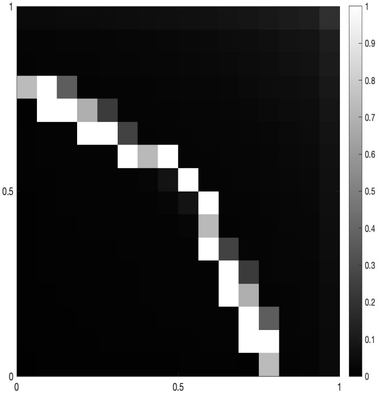
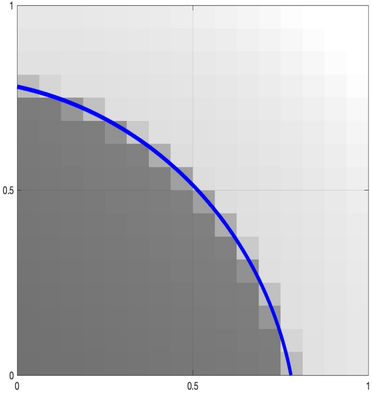


Error original-reconstructed

Example 3. Nonlinear techniques



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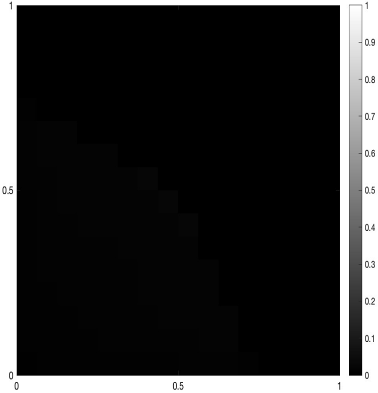

Error in non affected by disc

Conclusions and future work

- While linear interpolation may perform adequately in smooth regions, it introduces significant errors or overshoots near jumps or steep gradients.
- The Hermite-based nonlinear approach maintains smoothness while adapting to local variations, making it well-suited for data with discontinuities
- We can not avoid the stair effect.
- Detect where the discontinuity is and the values (x_i, y_j) where we obtain a good approximation and only use these points (with rectangles or triangles).
- Detect where the discontinuity is and extend information from the $H_{i,j}(x, y)$ well calculated to the cells with the discontinuities.

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