Adaptive methods for problems with infinitely many parameters and their computational complexity

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Nonlinear Approximation for High-Dimensional Problems

Workshop in honour of Albert Cohen

2016

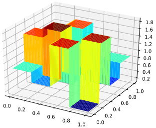
Parameter-dependent PDEs: Find $u = u(a) \in V$ such that $\mathcal{P}(a; u) = 0, \; a \in \mathcal{A}$

Elliptic model problem: $u \in V = H_0^1(D)$, $D \subset \mathbb{R}^d$, such that

$$-\nabla \cdot (a \nabla u) = f \text{ in } D, \quad u = 0 \text{ on } \partial D$$

- Model order reduction:

  efficient approximation of $a \mapsto u(a)$



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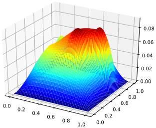
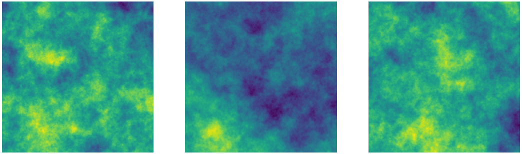


a $\qquad$ u

- Uncertainty quantification:

  probability measure on $\mathcal{A}$ modelling uncertainty in a, extract information on distribution of $u(a)$



Coefficient parametrizations: for $y \in Y$, find $u(y) \in V = H_0^1(D)$ such that

$$\int_D a(y)\nabla u(y) \cdot \nabla v \, dx = \int_D f v \, dx \quad \forall v \in V$$

- Piecewise constant model case: with partition $\{D_i\}$ of D, for $y \in Y = [-1,1]^P$,

$$a(y) = 1 + \theta \sum_{i=1}^{P} y_i \, \chi_{D_i}, \quad \theta \in (0,1)$$

- Affine parametrization with $y \in Y = [-1,1]^{\mathbb{N}}$,

$$a(y) = \bar{a} + \sum_{j=1}^{\infty} y_j \psi_j, \quad \bar{a}, \psi_j \in L^\infty(D)$$

  such that (uniform ellipticity): $0 < r \le a(y) \le R < \infty$ in D for all $y \in Y$.

- Lognormal coefficients: with $Y = \mathbb{R}^{\mathbb{N}}$,

$$a(y) = \exp\Big(\sum_{j\in\mathbb{N}} y_j \psi_j\Big), \quad y_j \sim \mathcal{N}(0,1),\ \psi_j \in L^\infty(D)$$

Aim: efficient approximations of $Y \ni y \mapsto u(y) \in V = H_0^1(D)$

A. Cohen and R. DeVore, *Approximation of high-dimensional parametric PDEs*, Acta Numerica, 2015.

Separation of variables

Rank-n expansions of parameter-dependent solution u,

$$u(y) \approx u_n(y) = \sum_{j=1}^{n} v_j\,\phi_j(y)\,, \quad v_j \in V,\ \phi_j : Y \to \mathbb{R}$$

- Reduced basis methods: solution snapshots $v_j := u(y^j)$, with $\phi_j(y)$ determined implicitly by Galerkin projection
- Approximation in $L^\infty(Y,V)$: Kolmogorov n-widths of $u(Y) \subset V$,

$$d_n\big(u(Y)\big)_V := \inf_{\substack{V_n \subset V \\ \dim(V_n)=n}} \sup_{y\in Y} \min_{v\in V_n} \|u(y) - v\|_V$$

- Controlling errors in $L^\infty(Y,V)$ problematic for high-dimensional Y

- Approximation in $L^2(Y,V,\mu)$, μ probability measure: Hilbert-Schmidt decomposition / SVD,

$$u = \sum_{j=1}^{n} \sigma_j \, \hat v_j \otimes \hat\phi_j, \quad \{\hat v_j\},\ \{\hat\phi_j\} \text{ orthonormal,}$$

best approximation by truncation, where

$$\sqrt{\sum_{j>n} \sigma_j^2} \le d_n(u(Y))_V .$$

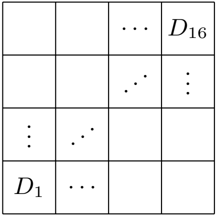
- Upper bounds for σ_j by prescribing $\hat\phi_j$, e.g. product orthonormal polynomial expansions in $L^2(Y,V,\mu)$: with $\mathcal{I} = \{1,\dots,P\}$ or $\mathcal{I} = \mathbb{N}$,

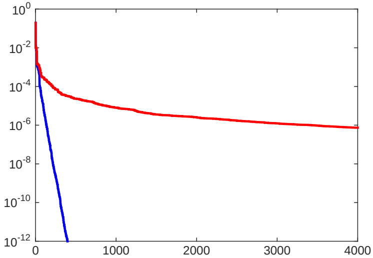
$$u(x,y) \approx \sum_{\nu\in\Lambda\subset\mathbb{N}_0^{\mathcal{I}}} u_\nu(x)\, L_\nu(y), \qquad L_\nu(y) := \prod_{i\in\mathcal{I}} L_{\nu_i}(y_i),$$

then $\sigma_j \le \|u_{\nu_j^*}\|_V$ with decreasing rearrangement $\|u_{\nu_j^*}\|_V$

Piecewise constant a on partition $\{D_i\}$, with $\bar{a} := 1$:

$$a(y) = 1 + \sum_{i=1}^{P} y_i \, \psi_i \,, \qquad \psi_i := \theta \chi_{D_i} \,, \quad \theta < 1 \,.$$





red: ordered norms $\|u_\nu\|_V$ of Legendre coefficients in $u(y) = \sum_\nu u_\nu L_\nu(y)$,

blue: singular values σ_j in SVD $u(y) = \sum_j \sigma_j \hat{v}_j \, \hat{\phi}_j(y)$

Upper bounds for Kolmogorov widths (B., Cohen '17):

recombining linearly dependent terms in Taylor polynomial expansions in y

- Trivial: $d_n(u(Y)) \lesssim \exp\big(-|\ln\theta| n^{-1/P}\big)$
- For piecewise constant parameters: when $\sum_{j=1}^{P} \psi_j = \theta\bar{a}$,

$$d_n(u(Y)) \lesssim \exp\big(-|\ln\theta| n^{-1/(P-1)}\big)$$

- Using further spatial symmetries:

D_3	D_4
D_1	D_2

$P = 4$ with regular 2×2 checkerboard. Then for any $f \in V'$,

$$d_n\big(u(Y)\big)_V \leq C \exp\Big(-\frac{|\ln\theta|}{8} n\Big).$$

($\rightsquigarrow$ Autio, Hannukainen '25)

M. Bachmayr and A. Cohen, *Kolmogorov widths and low-rank approximations of parametric elliptic PDEs*, Math Comp, 2017

Affinely parametrized linear elliptic PDEs

Parametric diffusion problem: for $y \in Y = [-1,1]^{\mathbb{N}}$, find $u(y) \in V = H_0^1(D)$ such that

$$\int_D a(y)\nabla u(y) \cdot \nabla v \, dx = \langle f, v \rangle, \quad \forall v \in V,$$

$$\text{where} \quad a(y) = \bar{a} + \sum_{j=1}^{\infty} y_j \psi_j, \quad \bar{a}, \psi_j \in L^\infty(D)$$

Uniform ellipticity assumption:

$$0 < r \leq a(y) \leq R < \infty\,, \quad \text{in } D, \text{ for all } y \in Y.$$

Here: for an $r > 0$,

$$\sum_{j \geq 1} |\psi_j| \leq \bar{a} - r. \qquad \text{(UEA)}$$

Objective: Approximate u in $L^\infty(Y, V)$ or $L^2(Y, V, \mu)$, with μ uniform measure on Y.

$$\mathcal{F} := \{\nu \in \mathbb{N}_0^{\mathbb{N}} \colon \nu \text{ has finitely many nonzero entries}\}, \quad |\nu| := \sum_{j\geq 1} \nu_j, \quad \nu! := \prod_{j\geq 1} \nu_j!$$

Taylor expansion: $$u = \sum_{\nu\in\mathcal{F}} t_\nu\, y^\nu \quad \text{with } y^\nu = \prod_{j\geq 1} y_j^{\nu_j} \text{ and } t_\nu = \frac{1}{\nu!}\partial^\nu u(0) \in V$$

Legendre expansion:

$$u = \sum_{\nu\in\mathcal{F}} u_\nu\, L_\nu(y) \quad \text{with orthonormal basis } \Big\{L_\nu(y) := \prod_{j\geq 1} L_{\nu_j}(y_j)\Big\}_{\nu\in\mathcal{F}} \text{ of } L^2(Y,V,\mu)$$

Theorem (Cohen, DeVore, Schwab '11).
Assume that (UEA) holds and $(\|\psi_j\|_{L^\infty})_{j\geq 1} \in \ell^p(\mathbb{N})$ for a $p \in (0,1)$, then $(\|t_\nu\|_V)_{\nu\in\mathcal{F}}$ and $(\|u_\nu\|_V)_{\nu\in\mathcal{F}}$ belong to $\ell^p(\mathcal{F})$.

Best n-term approximation: Take $\Lambda_{\mathsf{T},n}, \Lambda_{\mathsf{L},n} \subset \mathcal{F}$ corresponding to n largest coefficients,

$$\sup_y \Big\| u(y) - \sum_{\nu\in\Lambda_{\mathsf{T},n}} t_\nu\, y^\nu \Big\|_V \leq C n^{-\frac{1}{p}+1}, \quad \Big\| u - \sum_{\nu\in\Lambda_{\mathsf{L},n}} u_\nu\, L_\nu \Big\|_{L^2(U,V,\mu)} \leq C n^{-\frac{1}{p}+\frac{1}{2}}.$$

A. Cohen, R. DeVore, and Ch. Schwab, *Analytic regularity and polynomial approximation of parametric and stochastic elliptic PDE's*, Analysis and Applications, 2011.

Basic idea: improved results for ψ_j with spatial localization, still with basic assumption

$$\sum_{j\geq 1} |\psi_j| \leq \bar{a} - r. \tag{UEA}$$

Theorem (B., Cohen, Migliorati '17).
Let (UEA) hold and with $\rho_j > 1$, $j \in \mathbb{N}$, let

$$\sum_{j\geq 1} \rho_j |\psi_j| \leq \bar{a} - s \quad \text{for some } s > 0. \tag{UEA*}$$

Then

$$\sum_{\nu\in\mathcal{F}} \rho^{2\nu} \|t_\nu\|_V^2 < \infty, \qquad \sum_{\nu\in\mathcal{F}} \Big(\prod_{j\geq 1} (2\nu_j + 1) \Big)^{-1} \rho^{2\nu} \|u_\nu\|_V^2.$$

Corollary. Let $0 < p < 2$ and assume that for $q = q(p) := \frac{2p}{2-p}$, there exists a sequence $\rho = (\rho_j)_{j\geq 1}$ with $\rho_j > 1$ satisfying (UEA*) and $\big(\rho_j^{-1}\big)_{j\geq 1} \in \ell^q(\mathbb{N})$. Then $\big(\|t_\nu\|_V\big)_{\nu\in\mathcal{F}}$ and $\big(\|u_\nu\|_V\big)_{\nu\in\mathcal{F}}$ belong to $\ell^p(\mathcal{F})$.

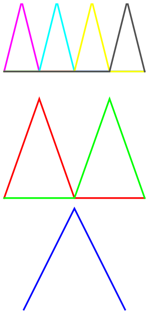
M. Bachmayr, A. Cohen, and G. Migliorati, *Sparse polynomial approximation of parametric elliptic PDEs. Part I: affine coefficients*, ESAIM M2AN, 2017

$y = (y_{\ell,m})_{\ell,m}$ with $y_{\ell,m} \sim \mathcal{U}(-1,1)$ i.i.d., and a with affine parameterization,

$$a(y) = a_0 + \sum_{\ell,m} y_{\ell,m} \psi_{\ell,m},$$

where $\sup_{x \in D} \sum_m \left|\psi_{\ell,m}(x)\right| \lesssim 2^{-\alpha\ell}$ for all $\ell \geq 0$

$\rightsquigarrow$ choose weights with $\rho_{\ell,m} \eqsim 2^{\beta\ell}$ with $\beta < \alpha$



Convergence of product Legendre expansions

Take $\Lambda_n \subset \mathcal{F}$ as indices of n largest $\|u_\nu\|_V$ in the expansion $u = \sum_{\nu \in \mathcal{F}} u_\nu L_\nu$. Then

$$\left\| u - \sum_{\nu \in \Lambda_n} u_\nu L_\nu \right\|_{L^2(Y,V,\mu)} \lesssim n^{-s} \quad \text{for any } s < \frac{\alpha}{d}$$

Lognormal coefficients: $a(y) = \exp\Big(\sum_{j\in\mathbb{N}} y_j\psi_j\Big)$, i.i.d. $y_j \sim \mathcal{N}(0,1)$, $\psi_j \in L^\infty(D)$

- product Hermite polynomial expansion $u(y) = \sum_{\nu\in\mathcal{F}} u_\nu H_\nu(y) \approx \sum_{\nu\in\Lambda\subset\mathcal{F}} u_\nu H_\nu(y)$

  where $u_\nu \in V$, $H_\nu(y) = \prod_{j\geq 1} H_{\nu_j}(y_j)$ with univariate Hermite polynomials H_{ν_j}

Theorem (B., Cohen, DeVore, Migliorati '17). Let $0 < q < \infty$ and $0 < p < 2$ such that $\frac{1}{q} = \frac{1}{p} - \frac{1}{2}$. Assume there exists a positive sequence $\rho = (\rho_j)_{j\geq 1}$ such that

$$(\rho_j^{-1})_{j\geq 1} \in \ell^q(\mathbb{N}) \quad \text{und} \quad \sup_{x\in D} \sum_{j\geq 1} \rho_j |\psi_j(x)| < \infty.$$

Then $\big(\|u_\nu\|_V\big)_{\nu\in\mathcal{F}} \in \ell^p(\mathcal{F})$.

- For $\{\psi_j\}$ with multilevel structure such that $\|\psi_j\|_{L^\infty} \lesssim 2^{-\alpha\ell(j)}$,

$$\Big\| u - \sum_{\nu\in\Lambda_n} u_\nu H_\nu \Big\|_{L^2(\mathbb{R}^\mathbb{N}, V, \bigotimes_{j\geq 1}\mathcal{N}(0,1))} \lesssim n^{-s} \quad \text{for any } s < \frac{\alpha}{d}$$

M. Bachmayr, A. Cohen, R. DeVore, and G. Migliorati, *Sparse polynomial approximation of parametric elliptic PDEs. Part II: lognormal coefficients*, ESAIM M2AN, 2017

Gaussian random fields

$D \subset \mathbb{R}^d$, centered Gaussian random field $\big(b(x)\big)_{x\in D}$ with covariance function

$$\mathbb{E}\big(b(x)b(x')\big) = K(x,x'), \quad x, x' \in D.$$

Given K, find $\{\psi_j\}$ such that $\qquad b(x) = \sum_{j=1}^{\infty} y_j \psi_j(x), \quad y_j \sim \mathcal{N}(0,1)$ i.i.d.

- Classical choice: Karhunen-Loève decomposition,

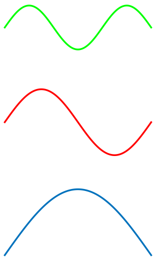
$$b(x) = \sum_{j=1}^{\infty} \sqrt{\lambda_j}\varphi_j(x)\, y_j \quad \text{with } y_j \sim \mathcal{N}(0,1) \text{ i.i.d.}$$

  with (λ_j, φ_j) eigenpairs of covariance operator, where φ_j is L^2-orthonormal

- Not the only option! Precise criterion (Luschgy, Pagès '09):

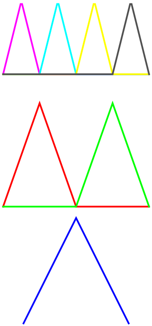
  ψ_j provide an expansion with y_j i.i.d. precisely when ψ_j Parseval frame in reproducing kernel Hilbert space of K

Expansions of the Brownian bridge



$K(s,t) = \min\{s,t\} - st$, with RKHS $H_0^1(0,1)$,
series $b = \sum_{j \geq 1} y_j \psi_j$ on $D = (0,1)$:

- **KL expansion:** $\psi_j(x) = \dfrac{\sqrt{2}}{\pi j} \sin(\pi j x)$,
  $\|\psi_j\|_{L^\infty} \sim j^{-1}$ with $|\mathrm{supp}\, \psi_j| = 1$.



- **Lévy-Ciesielski representation:**
  using Schauder basis (primitives of Haar system)

$$\psi_{\ell,m}(x) := 2^{-\ell/2} \psi(2^\ell x - m), \quad m = 0, \dots, 2^\ell - 1, \ \ell \geq$$

where $\psi(x) := \frac{1}{2}\big(1 - |2x - 1|\big)_+$.
Ordering from coarse to fine, $\psi_j := \psi_{\ell,m}$ for $j = 2^\ell + m$,
$\|\psi_j\|_{L^\infty} \sim j^{-\frac{1}{2}}$ and $|\mathrm{supp}\, \psi_j| \sim j^{-1}$.

$D \subset \mathbb{R}^d$, centered and stationary Gaussian random field $\big(b(x)\big)_{x\in D}$ with covariance function

$$\mathbb{E}\big(b(x)b(x')\big) = K(x,x') = k(x-x'), \quad x,x' \in D.$$

- **Matérn covariances**

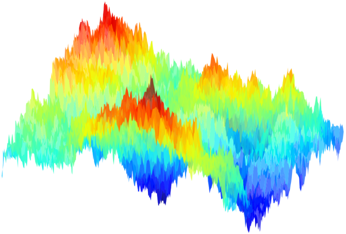
$$k(x) = \frac{2^{1-\nu}}{\Gamma(\nu)} \left(\frac{\sqrt{2\nu}|x|}{\lambda}\right)^{\nu} K_\nu\left(\frac{\sqrt{2\nu}|x|}{\lambda}\right), \quad \nu,\lambda > 0,$$

where K_ν is the modified Bessel function of the second kind, Fourier transform:

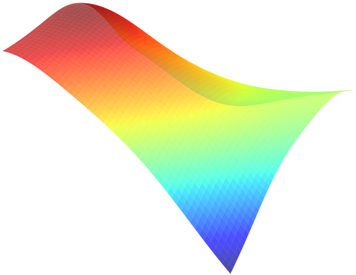
$$\hat{k}(\omega) = c_{\nu,\lambda}\left(\frac{2\nu}{\lambda^2} + |\omega|^2\right)^{-(\nu+d/2)}, \quad c_{\nu,\lambda} := \frac{2^d \pi^{d/2}\Gamma(\nu+d/2)(2\nu)^\nu}{\Gamma(\nu)\lambda^{2\nu}}.$$

(Exponential covariance $\nu = \frac{1}{2}$, Gaussian covariance $\nu \to \infty$)

Matérn samples



$\nu = \frac{1}{2}$



$\nu = 4$

Periodization of stationary Gaussian random fields

- Stationary periodic Gaussian random fields on a torus $\mathbb{T}$ with periodic covariance function: KL eigenfunctions are Fourier exponentials φ_j^{p} on $\mathbb{T}$
- **Periodization** (B., Cohen, Migliorati '17): periodize k with suitable cutoff function ϕ,

$$k_{\mathrm{p}}(x) := \sum_{n\in\mathbb{Z}^d} (k\phi)(x + 2\gamma n),$$

  positive semidefinite for sufficiently large γ

  if $$(1 + |\omega|^2)^{-s} \lesssim \widehat{k}(\omega) \lesssim (1 + |\omega|^2)^{-r}, \quad 0 < r \leq s$$

  and $$\lim_{R\to\infty} \int_{|x|>R} |\partial^\alpha k|\, dx = 0 \quad \text{for } |\alpha| \leq 2\lceil s\rceil,$$

  in particular all Matérn covariances[1]

  (Related results in special cases: Stein '02, Gneiting et al. '06, Helgason et al. '14, . . .)

- Leads to an improved version of sampling by circulant embedding[2]

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[1] M. Bachmayr, A. Cohen, and G. Migliorati, *Representations of Gaussian random fields and approximation of elliptic PDEs with lognormal coefficients*, JFAA, 2018

[2] M. Bachmayr, I. G. Graham, V. K. Nguyen, and R. Scheichl, *Unified analysis of periodization-based sampling methods for Matérn covariances*, SINUM, 2020

- Given: centered stationary Gaussian random field on domain D with covariance function k
- Embed D into a torus $\mathbb{T}$, periodized random field with covariance k_{p}
- Start from periodic $L^2(\mathbb{T})$-orthonormal Meyer wavelets

$$\Psi_{\ell,m} = \sum_j c_j^{(\ell,m)} \varphi_j^{\mathrm{p}},$$

with localized supports on $\mathbb{T}$.

- Apply square root of the covariance operator on $\mathbb{T}$,

$$\psi_{\ell,m}^{\mathrm{p}} = \sum_j \sqrt{\lambda_j^{\mathrm{p}}}\, c_j^{(\ell,m)} \varphi_j^{\mathrm{p}},$$

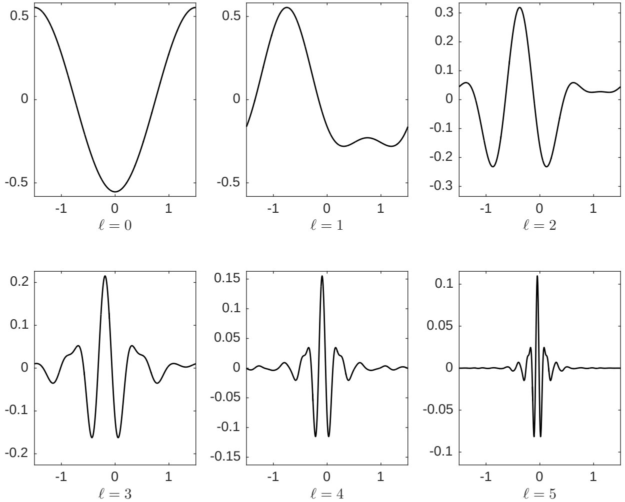
$\psi_{\ell,m} := \psi_{\ell,m}^{\mathrm{p}}|_D$ Parseval frame of the reproducing kernel Hilbert space of k.

- Verify that also $\psi_{\ell,m}$ are still localized, under additional assumptions on $\widehat{k}$ satisfied by Matérn covariance (decay of higher-order derivatives of $\widehat{k\phi}^{1/2}$)

M. Bachmayr, A. Cohen, and G. Migliorati, *Representations of Gaussian random fields and approximation of elliptic PDEs with lognormal coefficients*, JFAA, 2018

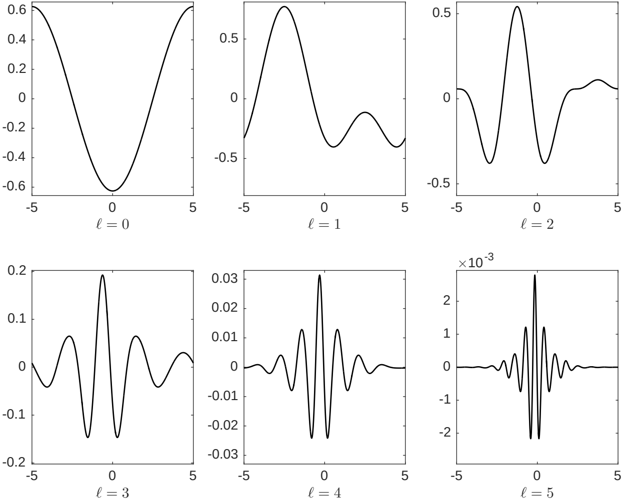
Matérn wavelets, 1D case on $D = [-\frac{1}{2}, \frac{1}{2}]$

Matérn covariance with $\lambda = 1$, $\nu = \frac{1}{2}$: plots of ψ_ℓ, where $\psi_{\ell,m}(x) = \psi_\ell(2^\ell x - m)$



Matérn wavelets, 1D case on $D = [-\frac{1}{2}, \frac{1}{2}]$

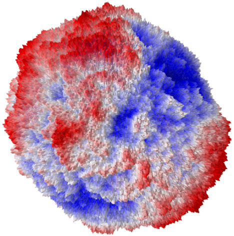
Matérn covariance with $\lambda = 1$, $\nu = 4$: plots of ψ_ℓ, where $\psi_{\ell,m}(x) = \psi_\ell(2^\ell x - m)$



Conclusion: For $a = \exp(b)$, Matérn-type b with realizations in $C^{0,\beta}(\overline{D})$ for $\beta < \alpha$ in wavelet representation, where $\|\psi_j\|_{L^\infty} \lesssim 2^{-\alpha \ell(j)}$,

$$\Big\| u - \sum_{\nu \in \Lambda_n} u_\nu H_\nu \Big\|_{L^2(\mathbb{R}^\mathbb{N}, V, \bigotimes_{j\in\mathbb{N}} \mathcal{N}(0,1))} \lesssim n^{-s} \quad \text{for any } s < \frac{\alpha}{d}$$

- Analogous representations for isotropic random fields on the sphere[3]: based on spherical needlets (Narcowich, Petrushev, Ward '06)
- Work in progress: more general smooth surfaces



[3] M. Bachmayr and A. Djurdjevac, *Multilevel representations of isotropic Gaussian random fields on the sphere*, IMA JNA, 2022

Fully discrete approximability

For each $\nu \in \mathcal{F}$, choose $V_\nu \subset V$ with $N_\nu := \dim V_\nu < \infty$ and take approximations u_N from

$$\mathcal{V}_N = \Big\{ \sum_{\nu\in\mathcal{F}} v_\nu L_\nu \colon v_\nu \in V_\nu \Big\}, \qquad N = \sum_{\nu\in\mathcal{F}} N_\nu$$

For affine case: $a(y) = \bar a + \sum_{\ell,m} y_{\ell,m}\psi_{\ell,m}$ uniformly elliptic, $Y \simeq [-1,1]^{\mathbb{N}}$

Adaptive approximations ($d \geq 2$) (B., Cohen, Dũng, Schwab '17)

Let $d \geq 2$ and $\alpha \in (0,1]$, let a be given in multilevel expansion with

$$\sup_D \sum_m |\psi_{\ell,m}| \lesssim 2^{-\alpha\ell}, \quad \sup_D \sum_m |\nabla\psi_{\ell,m}| \lesssim 2^{-(\alpha-1)\ell} \quad \text{for all } \ell \geq 0,$$

let D be convex or smooth and let $f \in L^2(D)$. Then for each N there exist $(V_\nu)_{\nu\in\mathcal{F}}$ such that for the corresponding $\mathcal{V}_N$,

$$\inf_{u_N\in\mathcal{V}_N} \|u - u_N\|_{L^2(Y,V,\mu)} \lesssim N^{-s} \quad \text{for any } s < \frac{\alpha}{d}.$$

M. Bachmayr, A. Cohen, D. Dũng, and Ch. Schwab, *Fully discrete approximation of parametric and stochastic elliptic PDEs*, SINUM, 2017

- How to choose $(V_\nu)_{\nu\in\mathcal{F}}$, total number of degrees of freedom $N = \sum_{\nu\in\mathcal{F}} N_\nu$?

- Adaptive wavelet approximation for each ν:

  $\{\Psi_\lambda\}_{\lambda\in\mathcal{S}}$ wavelet Riesz basis of $V = H_0^1(D)$,

$$\Big\|\sum_{\lambda,\nu} \mathbf{v}_{\lambda,\nu}\Psi_\lambda \otimes L_\nu\Big\|^2_{L^2(Y,V,\mu)} \eqsim \sum_{\lambda,\nu}|\mathbf{v}_{\lambda,\nu}|^2, \quad \mathbf{v}\in\ell^2(\mathcal{S}\times\mathcal{F})$$

$$\rightsquigarrow \text{ expansion } \quad u = \sum_{\lambda,\nu} \mathbf{u}_{\lambda,\nu}\Psi_\lambda \otimes L_\nu$$

- Best N-term approximation by keeping (λ,ν) with N largest $|\mathbf{u}_{\lambda,\nu}|$:

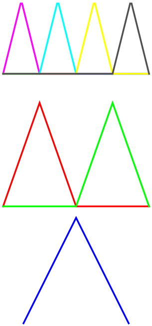
$$\|u - u_{[N]}\|_{L^2(Y,V,\mu)} \eqsim \|\mathbf{u}-\mathbf{u}_{[N]}\|_{\ell_2} \le N^{-s}\|\mathbf{u}\|_{\mathcal{A}^s} \qquad \rightsquigarrow \qquad N(\varepsilon) = \|\mathbf{u}\|_{\mathcal{A}^s}^{\frac{1}{s}}\varepsilon^{-\frac{1}{s}}$$

Example

Multiscale representation in $d = 1$, with $\alpha = 1$,

$$\psi_{\ell,m}(x) := c2^{-\ell}\psi(2^\ell x - m)$$

$$a(y) = 1 + \sum_{\ell,m} y_{\ell,m}\psi_{\ell,m} \quad \rightsquigarrow \quad u(y) = \sum_{\nu\in\mathcal{F}} u_\nu L_\nu(y)$$

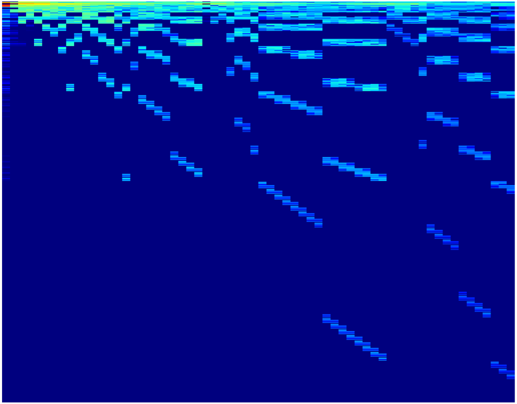


$d = 1$: $\quad a(y) = \bar{a} + \sum_{j \geq 1} y_j \, \psi_j , \;\; \psi_j$ hierarchical hat functions, $\|\psi_j\|_{L^\infty} \lesssim 2^{-\alpha \ell(j)}$

Values $|\mathbf{u}_{\mu,\nu}|$ (for $\alpha = 1$):

$\xrightarrow{\nu}$ (decreasing $\|u_\nu\|_V$)

$\mu \downarrow$



Stochastic Galerkin discretization: $u_N \in \mathcal{V}_N$ such that

$$\int_{[-1,1]^{\mathbb{N}}} \int_D a\nabla u_N \cdot \nabla v \, dx \, d\mu(y) = \int_{[-1,1]^{\mathbb{N}}} \langle f, v \rangle \, d\mu(y), \quad \text{for all } v \in \mathcal{V}_N$$

Operator representation w.r.t. spatial-parametric Riesz basis $\{\Psi_\lambda \otimes L_\nu\}_{\lambda\in\mathcal{S},\nu\in\mathcal{F}}$,

$$\mathbf{A} = \sum_{j\geq 0} \mathbf{A}_j \otimes \mathbf{M}_j : \ell^2(\mathcal{S}\times\mathcal{F}) \to \ell^2(\mathcal{S}\times\mathcal{F})$$

where

$$\mathbf{A}_0 = \left(\int_D \bar{a}\nabla\Psi_{\lambda'} \cdot \nabla\Psi_\lambda\right)_{\lambda,\lambda'\in\mathcal{S}}, \quad \mathbf{M}_0 = \left(\delta_{\nu,\nu'}\right)_{\nu,\nu'\in\mathcal{F}}$$

$$\mathbf{A}_j = \left(\int_D \psi_j \nabla\Psi_{\lambda'} \cdot \nabla\Psi_\lambda\right)_{\lambda,\lambda'\in\mathcal{S}}, \quad \mathbf{M}_j = \left(\int_U y_j L_\nu(y) L_{\nu'}(y)\, d\mu(y)\right)_{\nu,\nu'\in\mathcal{F}}, \quad j \geq 1.$$

$\rightsquigarrow$ well-conditioned sequence-space formulation $\mathbf{A}\mathbf{u} = \mathbf{f}$.

Standard adaptive Galerkin scheme

(Cohen, Dahmen, DeVore '01; Gantumur, Harbrecht, Stevenson '07)

Given $\Lambda^k \subset \mathcal{S}\times\mathcal{F}$, compute Galerkin solution $\mathbf{u}_k$ on Λ^k, approximate $\mathbf{r}_k = \mathbf{A}\mathbf{u}_k - \mathbf{f}$, and with fixed $\mu \in (0,1)$ set

$$\Lambda^{k+1} = \Lambda^k \cup \hat{\Lambda} \quad \text{with } \hat{\Lambda} \text{ of minimal size such that } \|\mathbf{r}|_{\hat{\Lambda}}\|_{\ell^2} \geq \mu\|\mathbf{r}\|_{\ell^2}$$

Direct residual approximation

- Residual approximation for stochastic Galerkin systems can be done based on standard compression techniques for $\mathbf{A}$ (using s^*-compressibility)
- For ψ_j with global supports, rates generally not optimal (Gittelson '13, '14)
- Observation[4] for $\{\psi_j\}$ with multilevel structure such that $\|\psi_j\|_{L^\infty} \lesssim 2^{-\alpha\ell(j)}$ (ordered by level): $\mathbf{A} = \sum_{j\geq 0} \mathbf{A}_j \otimes \mathbf{M}_j$ satisfies

$$\Big\| \sum_{j>M} \mathbf{A}_j \otimes \mathbf{M}_j \Big\| \lesssim M^{-\frac{\alpha}{d}}.$$

- Compression based on approximations $\sum_{j\leq M} \mathbf{A}_j \otimes \mathbf{M}_j$ combined with spatial s^*-compressibility of the $\mathbf{A}_j$: sub-optimal rates

$$s^* = \frac{t}{t+d}\frac{\alpha}{d}$$

when $\psi_j \nabla \Psi_\lambda \in H^t$.

[4] M. Bachmayr, A. Cohen, and W. Dahmen, *Parametric PDEs: Sparse or low-rank approximations?*, IMA JNA, 2018

- Iteratively refined stochastic Galerkin discretizations with spatial approximation by H^2-regular spline wavelets, piecewise polynomial (approximations of) ψ_j

New residual approximation strategy:

- Adaptive semidiscrete operator compression in parametric variables, based on $\sum_{j\leq M} \mathbf{A}_j \otimes \mathbf{M}_j$,
- Spatial error estimation using tree index sets and piecewise polynomial structure without adaptive operator compression (Stevenson '14; Binev '18)

Optimality (B., Voulis '22)

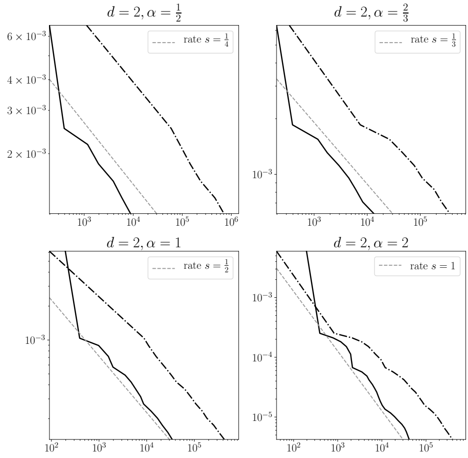
If the best approximation to u converges at rate $s < \frac{\alpha}{d}$ then for each $\varepsilon > 0$, the adaptive scheme with appropriately chosen parameters finds an approximation u_ε with $\|u - u_\varepsilon\|_\mathcal{V} \leq \varepsilon$ using $\mathcal{O}\big(1 + \varepsilon^{-\frac{1}{s}}(1 + |\log \varepsilon|)\big)$ operations.

(see also Bespalov, Praetorius, Ruggeri '21: optimal cardinality under saturation assumption)

M. Bachmayr and I. Voulis, *An adaptive stochastic Galerkin method based on multilevel expansions of random fields: Convergence and optimality*, ESAIM M2AN, 2022

Numerical experiments: wavelets, $d = 2$ (B., Voulis '22)

$D = (0,1)^2$, $\psi_{\ell,m}$ hierarchical piecewise linear hat functions with $\|\psi_{\ell,m}\|_{L_\infty} \lesssim 2^{-\alpha\ell}$, spatial discretization by C^1 piecewise polynomial DGH multiwavelets of order 6; expected fully discrete rate $\frac{\alpha}{2}$.



Residual estimates as a function of #dof (—) and of computation time (--)

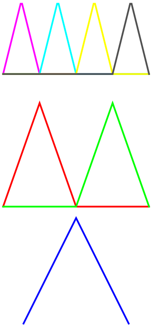
Finite element approximations in space?

Aim: $u(y) = \sum_{\nu\in\Lambda} u_\nu L_\nu(y)$ with $u_\nu \in \mathbb{P}_1(\mathcal{T}_\nu) \cap V$, separate mesh $\mathcal{T}_\nu$ for each ν

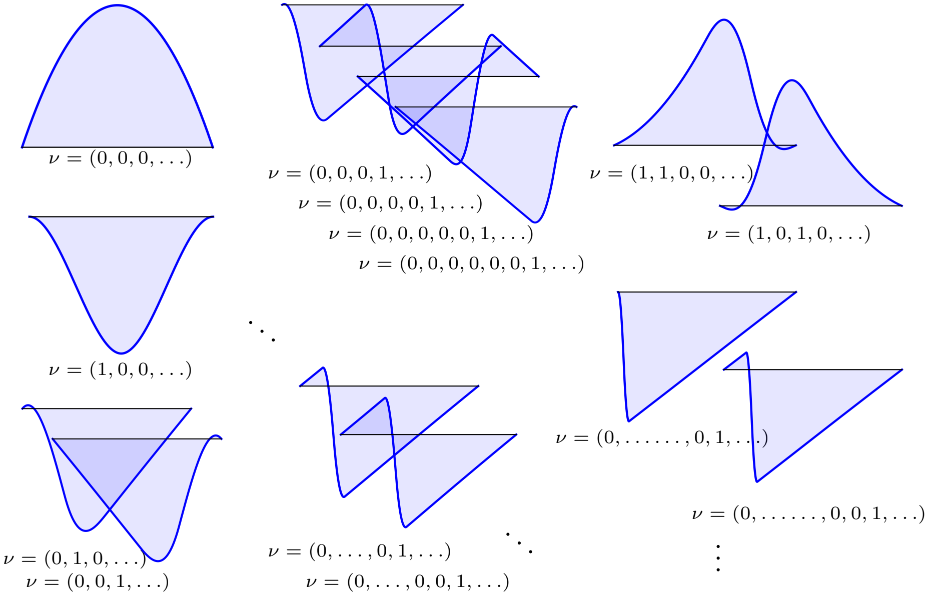
Same example with $d = 1$, $\alpha = 1$,

$$\psi_{\ell,m}(x) := c2^{-\ell}\psi(2^\ell x - m)$$

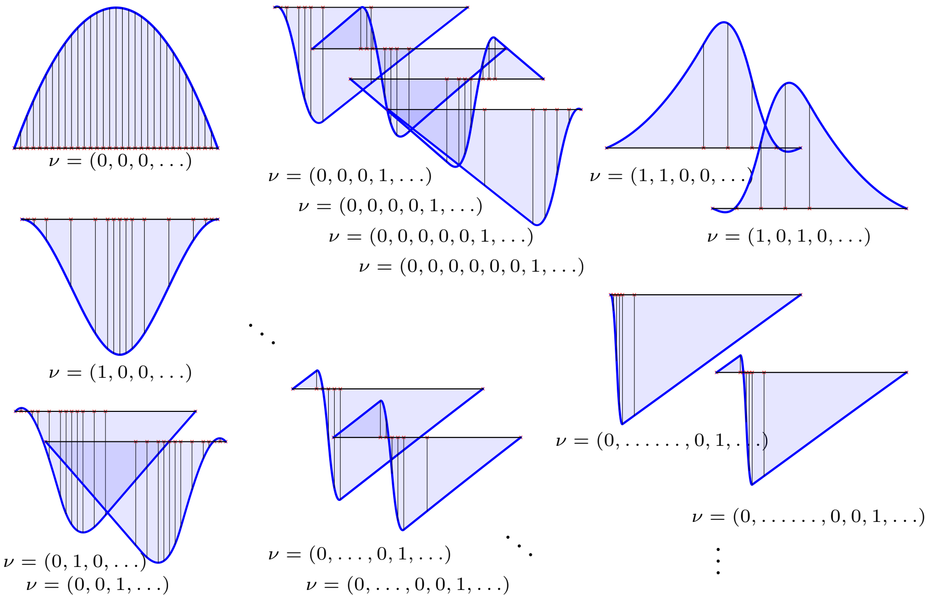
$$a(y) = 1 + \sum_{\ell,m} y_{\ell,m}\psi_{\ell,m} \quad \rightsquigarrow \quad u(y) = \sum_{\nu\in\mathcal{F}} u_\nu L_\nu(y)$$



Legendre coefficient functions $u_\nu \colon [0,1] \to \mathbb{R}$ in $u(y) = \sum_{\nu \in \mathcal{F}} u_\nu L_\nu(y)$ with diffusion coefficient a expanded in terms of hierarchical hat functions:



Best (dyadic) grids for piecewise linear approximations of u_ν:



Towards an optimal adaptive finite element solver

- Piecewise affine linear finite element approximation on independent adaptive mesh for each u_ν, refinement by standard newest vertex bisection
- Again using adaptive operator compression in the stochastic variables.
- Standard finite element error estimation strategies (e.g., residual estimators) not applicable due to interactions between meshes, lack of Galerkin orthogonality (see also Cohen, DeVore, Nochetto '12)
- Instead use BPX frame coefficients (cf. Harbrecht, Schneider '16): for $r \in V' = H^{-1}(D)$,

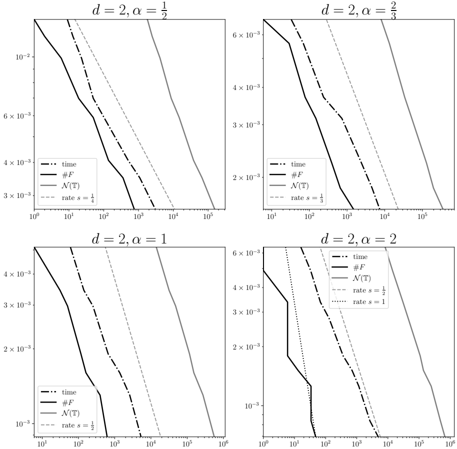
$$\|r\|_{V'}^2 \eqsim \sum_{j=0}^{\infty} \sum_{k \in \mathcal{N}_j} |\langle r, \varphi_{j,k} \rangle|^2$$

  with $\varphi_{j,k}$ piecewise linear hat function on level j (with $\|\varphi_{j,k}\|_{H_0^1(D)} \eqsim 1$)
- Choose refinements by tree-based selection of frame-based indicators (Binev '18)

First result[5]**:** reduction of stochastic Galerkin energy norm error by uniform factor in each step of the adaptive scheme, linear convergence to exact solution.

[5] M. Bachmayr, M. Eigel, H. Eisenmann and I. Voulis, *A convergent adaptive finite element stochastic Galerkin method based on multilevel expansions of random fields*, to appear in SINUM

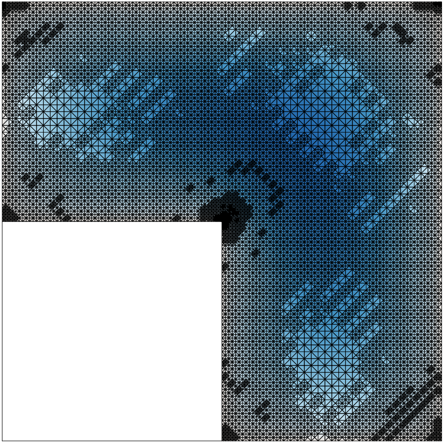
L-shaped domain, multilevel hat functions $\psi_{\ell,m}$ with $\|\psi_{\ell,m}\|_{L_\infty} \lesssim 2^{-\alpha\ell}$, spatial discretization by $\mathbb{P}_1$ elements on newest vertex bisection meshes.

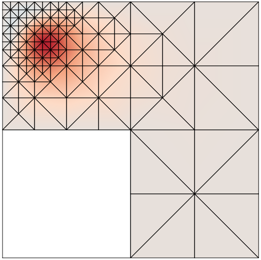


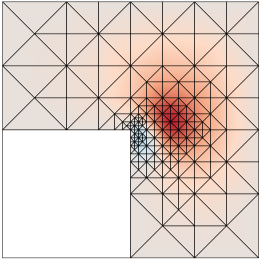
Residual estimates as a function of #dof (parametric —, all —) and of computation time (- ·)

Finite elements, $d = 2$

(B., Eigel, Eisenmann, Voulis '24)







Optimal complexity

First consider optimality of generated discretizations assuming

- affine coefficients $a(y) = \bar{a} + \sum_{j \geq 1} y_j \psi_j$,
- best approximations of u in $\mathcal{V}_N$ converging as $\mathcal{O}(N^{-s})$ with $s < \alpha/d$.

Theorem (B., Eisenmann, Voulis '25; abridged).

- The meshes generated by the method have optimal cardinality:

$$\|u - u_N\|_{L^2(Y,V,\mu)} \leq \varepsilon \quad \text{with} \quad N \lesssim \varepsilon^{-1/s}.$$

- If $\{\psi_j\}$ have multilevel structure, near-optimal total number of operations

$$\mathcal{O}\big(\varepsilon^{-1/s}(1 + |\log \varepsilon|^3)\big) \qquad \text{for all } s < \alpha/d.$$

- Main new ingredient: stability property of finite element frames on adaptively refined (newest vertex bisection) meshes

M. Bachmayr, H. Eisenmann and I. Voulis, *Adaptive stochastic Galerkin finite element methods: Optimality and non-affine coefficients*, arXiv:2503.18704

Extension to non-affine coefficients

- Uniformly elliptic coefficients of the form (e.g., log-uniform case $g = \exp$)

$$a(y) = g\Big(\sum_{j\geq 1} y_j\theta_j\Big) \quad \text{with i.i.d. } y_j \sim \mathcal{U}(-1,1),$$

- Requires new semi-discrete operator compression
- Basic strategy: for g analytic in sufficiently large rectangle in $\mathbb{C}$, use polynomial approximations of g.

Theorem (B., Eisenmann, Voulis '25; abridged).

Assuming $\{\psi_j\}$ with multilevel structure as before and best approximation rate $s < \alpha/d$, then

$$\|u - u_N\|_{L^2(Y,V,\mu)} \leq \varepsilon \quad \text{with} \quad N \lesssim \varepsilon^{-1/s}$$

using a number of operations of order

$$\mathcal{O}\big(\varepsilon^{-1/s'}(1 + |\log \varepsilon|^r)\big) \qquad \text{for all } s' < s < \alpha/d$$

with $r > 0$ independent of s', s, k.

M. Bachmayr, H. Eisenmann and I. Voulis, *Adaptive stochastic Galerkin finite element methods: Optimality and non-affine coefficients*, arXiv:2503.18704

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Happy birthday, Albert!

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