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The BBB formulation of evolution PDEs

Optimal transport

I ime

Harmonic time averaging

discretization

decomposition

equation

Easy path selection

Discretization of the Ballistic-Benamou-Brenier formulation of the porous medium equation

Jean-Marie Mirebeau

ENS Paris-Saclay, CNRS, University Paris-Saclay

July 5, 2025

Nonlinear approximation for High-Dimensional problems
Workshop in honor of Albert Cohen
In collaboration with E. Stampfli, Y. Brenier and T. Gallouet.

Erwan Stampfli, M, Discretization and convergence of the ballistic Benamou-Brenier formulation of the porous medium and Burgers' equations, preprint, 2025

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The BBB formulation of evolution

PDEs Optimal

transport

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Harmonic time averaging

Space discretization

Voronoi decomposition

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equation

Easy path

The BBB formulation of evolution PDEs Connection with optimal transport

Time discretization

Harmonic time averaging

Space discretization
Voronoi/Selling decomposition

Burgers equation
Easy path selection

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Discretization of the BBB

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The BBB

formulation of evolution PDEs Optimal transport

Time discretization Harmonic time averaging

Space discretization Voronoi

Burgers equation

selection

The quadratic porous medium equation (QPME) reads

$$\partial_t u = \frac{1}{2} \operatorname{div}(\mathcal{D} \nabla u^2), \qquad \qquad u(t=0) = u_0.$$

Solved over the time interval [0, T], domain $\mathbb{T}^d = (\mathbb{R}/\mathbb{Z})^d$, initial data $u_0 \geq 0$, smooth diffusion tensors $\mathcal{D} : \mathbb{T}^d \to \mathcal{S}_d^{++}$.

- Reformulation by Y. Brenier as a convex optimization problem in space and time, related with optimal transport.
- Unconventional numerical method, relying on space-time FFT and proximal operators. No CFL, 2nd order.
- ► Applies to various conservation equations, fluid mechanics.

Y. Brenier, Examples of hidden convexity in nonlinear PDEs. Book available online (2020).

D. Vorotnikov, Hidden convexity and <u>Dafermos' principle</u> for some dispersive equations. arXiv (2025) "The physical solution dissipates entropy earliest and fastest"

S. Singh, J. Ginster, A. Acharya, *A hidden convexity of nonlinear elasticity*. Journal of Elasticity (2024).

Mirebeau

- Non-linear PDEs often do not admit smooth solutions.
- ► Consider the weak form: $\forall \phi \in C_c^{\infty}([0, T[\times \mathbb{T}^d)])$

$$\int_{[0,T]\times\mathbb{T}^d} (\partial_t u - \frac{1}{2}\operatorname{div}(\mathcal{D}\nabla u^2))\phi = \int_{[0,T]\times\mathbb{T}^d} -(u - u_0)\partial_t \phi - \frac{1}{2}u^2\operatorname{div}(\mathcal{D}\nabla\phi)$$

formulation of evolution PDEs

Optimal transport

Harmonic time

Harmonic time averaging

discretization

Voronoi decomposition

decomposition

equation

Easy path selection

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The BBB formulation of evolution PDEs

Optimal transport

transpo

discretizatio

Harmonic time averaging

Space

discretization Voronoi

decomposition

Burger

Easy path

▶ Non-linear PDEs often do not admit smooth solutions.

► Consider the weak form: $\forall \phi \in C_c^{\infty}([0, T[\times \mathbb{T}^d)])$

$$0 = \int_{[0,T]\times\mathbb{T}^d} -(u-u_0)\partial_t \phi - \frac{1}{2}u^2 \operatorname{div}(\mathcal{D}\nabla\phi)$$

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The BBB formulation of evolution PDEs

Optimal transport

Time discretizat

Harmonic time averaging

discretization

Voronoi decomposition

equation

Easy path

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- Ex: Barenblatt profile (non-smooth, compact support)
- Issue: possibly several weak solutions.(Are there non-positive QPME solutions ?)

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The BBB formulation of evolution PDEs

Optimal transport

Time

Harmonic time

Space discretization

Voronoi decomposition

Burgers equation

Easy path selection ► Non-linear PDEs often do not admit smooth solutions.

► Consider the weak form: $\forall \phi \in C_c^{\infty}([0, T[\times \mathbb{T}^d)])$

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Ex: Barenblatt profile (non-smooth, compact support)

Issue: possibly several weak solutions.(Are there non-positive QPME solutions ?)

BBB selection principle: minimize the total kinetic energy $\frac{1}{2}\int_{[0,T]\times\mathbb{T}^d}u^2$ (or a suitable entropy) among all weak solutions

$$\inf_{u(0)=u_0} \sup_{\phi(T)=0} \int_{[0,T]\times\mathbb{T}^d} \frac{1}{2}u^2 + \underbrace{\left(\partial_t u - \frac{1}{2}\operatorname{div}(\mathcal{D}\nabla u^2)\right)\phi}_{\text{Understood in weak sense}}$$

Project the null function onto the manifold of weak solutions. (Acharya et al often project an initial guess $u_{\rm ref}$)

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The BBB formulation of evolution PDEs

Optimal

discretizat

Harmonic time averaging

Space discretization Voronoi

decomposition

equation

Easy path

▶ Non-linear PDEs often do not admit smooth solutions.

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$$\inf_{u(0)=u_0} \sup_{\phi(T)=0} \int_{[0,T]\times\mathbb{T}^d} \frac{1}{2}u^2 - (u-u_0)\partial_t \varphi - \frac{1}{2}u^2 \operatorname{div}(\mathcal{D}\nabla\varphi)$$

Project the null function onto the manifold of weak solutions. (Acharya et al often project an initial guess u_{ref})

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The BBB formulation of evolution PDEs

Optimal transport

Time discretizati

Harmonic time averaging

discretization Voronoi

decomposition

equation

Easy path selection ▶ Non-linear PDEs often do not admit smooth solutions.

▶ Consider the weak form: $\forall \phi \in C_c^{\infty}([0, T[\times \mathbb{T}^d)])$

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BBB selection principle: minimize the total kinetic energy $\frac{1}{2}\int_{[0,T]\times\mathbb{T}^d}u^2$ (or a suitable entropy) among all weak solutions

$$\inf_{\substack{u(0)=u_0\\ \phi(\mathcal{T})=0}} \sup_{\substack{\phi(\mathcal{T})=0\\ \phi(\mathcal{T})=0}} \int_{[0,\mathcal{T}]\times\mathbb{T}^d} \frac{1}{2}u^2 - (u-u_0)\partial_t \varphi - \frac{1}{2}u^2\operatorname{div}(\mathcal{D}\nabla\varphi)$$

$$\stackrel{?}{=} \sup_{\substack{\phi(\mathcal{T})=0\\ \phi(\mathcal{T})=0}} \inf_{\substack{u(0)=u_0\\ 0}} \int_{[0,\mathcal{T}]\times\mathbb{T}^d} \frac{1}{2}u^2 (1-\operatorname{div}(\mathcal{D}\nabla\phi)) - (u-u_0)\partial_t \phi.$$

Question mark: does duality hold? (We'll come back to this)

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The BBB formulation of evolution **PDEs**

Optimal transport

Harmonic time averaging

discretization

Voronoi

decomposition

Easy path selection

Dual formulation

Assuming duality holds, kinetic energy minimization reads

$$\sup_{\phi(T)=0}\inf_{u(0)=u_0}\int_{[0,T]\times\mathbb{T}^d}\tfrac{1}{2}u^2(1-\operatorname{div}(\mathcal{D}\nabla\phi))-(u-u_0)\partial_t\phi$$

Minimize pointwise w.r.t. u, assuming $div(\mathcal{D}\nabla\phi) < 1$:

$$u = \frac{\partial_t \phi}{1 - \mathsf{div}(\mathcal{D} \nabla \phi)}$$

Convex optimization problem w.r.t. ϕ

$$\inf_{\phi(T)=0} \int_{[0,T]\times\mathbb{T}^d} \frac{(\partial_t \phi)^2}{2(1-\operatorname{div}(\mathcal{D}\nabla\phi))} - u_0 \partial_t \phi.$$

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The BBB formulation of evolution

PDEs Optimal

transport

discretization

Harmonic time averaging

Space discretization

Voronoi decomposition

.

equatio

Easy path

Dual formulation

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Minimize pointwise w.r.t. u, assuming $div(\mathcal{D}\nabla\phi) < 1$:

$$\partial_t \phi = (1 - \operatorname{div}(\mathcal{D}\nabla\phi))u, \qquad \phi(T) = 0.$$

Convex optimization problem w.r.t. ϕ

$$\inf_{\phi(T)=0} \int_{[0,T]\times\mathbb{T}^d} \frac{(\partial_t \phi)^2}{2(1-\operatorname{div}(\mathcal{D}\nabla\phi))} - u_0 \partial_t \phi.$$

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The BBB formulation of evolution **PDEs**

Optimal

transport

Harmonic time averaging

discretization

Voronoi decomposition

Easy path selection

Dual formulation

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Minimize pointwise w.r.t. u, assuming $div(\mathcal{D}\nabla\phi) < 1$:

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Convex optimization problem w.r.t. ϕ

$$\inf_{\phi(T)=0} \int_{[0,T]\times\mathbb{T}^d} \frac{(\partial_t \phi)^2}{2(1-\operatorname{div}(\mathcal{D}\nabla\phi))} - u_0 \partial_t \phi.$$

Ratio understood as the convex l.s.c. perspective function

$$\mathcal{P}(a,b) := egin{cases} a^2/(2b) & ext{if } b>0, \ 0 & ext{if } a=b=0, \ \infty & ext{else}. \end{cases}$$

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The BBB formulation of evolution PDEs

Optimal transport

Time

Harmonic time

Space discretization

Voronoi

decomposition

Burger

Easy path

Dual formulation

Assuming duality holds, kinetic energy minimization reads

$$\sup_{\phi(T)=0}\inf_{u(0)=u_0}\int_{[0,T]\times\mathbb{T}^d}\tfrac{1}{2}u^2(1-\operatorname{div}(\mathcal{D}\nabla\phi))-(u-u_0)\partial_t\phi$$

▶ Minimize pointwise w.r.t. u, assuming $div(\mathcal{D}\nabla\phi) < 1$:

$$\partial_t \phi = (1 - \operatorname{div}(\mathcal{D}\nabla\phi))u, \qquad \quad \phi(T) = 0.$$

Convex optimization problem w.r.t. ϕ

$$\inf_{\phi(T)=0} \int_{[0,T]\times\mathbb{T}^d} \frac{(\partial_t \phi)^2}{2(1-\operatorname{div}(\mathcal{D}\nabla\phi))} - u_0 \partial_t \phi.$$

▶ Letting $m := \partial_t \phi$ and $\rho := 1 - \text{div}(\mathcal{D}\nabla \phi)$ we find

$$\inf_{m,\rho} \int_{[0,T] \times \mathbb{T}^d} \frac{m^2}{2\rho} - u_0 m, \quad \text{s.t. } \partial_t \rho + \operatorname{div}(\mathcal{D} \nabla m) = 0, \ \rho(T) = 1.$$

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The BBB formulation of evolution

Optimal transport

Time

Harmonic time averaging

Space discretization

Voronoi decomposition

Burgers

Easy path selection

A ballistic variant of optimal transport

The Benamou-Brenier formulation of optimal transport reads

$$\inf_{m,\rho} \int_{[0,1]\times\Omega} \frac{|m|^2}{2\rho}, \quad \text{s.t. } \partial_t \rho + \text{div } m = 0, \ \rho(0) = \rho_0, \rho(1) = \rho_1,$$

where ρ_0, ρ_1 are probability densities on Ω .

Similar structure as the QPME reformulation, with the caveats:

- ▶ Boundary conditions at both endpoints, vs $\rho(T) = 1$.
- First order continuity equation, vs $\partial_t \rho + \text{div}(\mathcal{D}\nabla m) = 0$.
- ▶ Absence of the first order term, vs $-u_0m$.

J.-D. Benamou, Y. Brenier, A computational fluid mechanics solution to the Monge-Kantorovich mass transfer problem. Numer Math (2000).

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The BBB formulation

Optimal

Time

discretization

Harmonic time averaging

Space discretization

Voronoi decomposition

equatio

Easy path selection

The BBB formulation of evolution PDEs Connection with optimal transport

Time discretization
Harmonic time averaging

Space discretization
Voronoi/Selling decomposition

Burgers equation
Easy path selection

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The BBB formulatio of evolution

of evolution PDEs
Optimal

transport

Time discretization

Harmonic time

Harmonic time averaging

discretization

Voronoi decomposition

Burgers

equation Easy path selection Staggered time grids, with half timestep $\tau > 0$, $T/(2\tau) \in \mathbb{N}$ $\mathcal{T}_{\tau} := \{0, 2\tau, \cdots, T\}, \quad \mathcal{T}_{\tau}' := \{\tau, 3\tau, \cdots, T - \tau\}.$

$$\partial_{\tau}u(t,x):=\frac{u(t+\tau,x)-u(t-\tau,x)}{2\tau}.$$

Spatial grid $\mathbb{T}_h := \{0, h, \cdots, 1-h\}$, where $1/h \in \mathbb{N}$.

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The BBB formulatio of evolution

Optimal transport

Time discretization

Harmonic time averaging

discretization Voronoi

decomposition

Burgers equation Easy path Staggered time grids, with half timestep $\tau > 0$, $T/(2\tau) \in \mathbb{N}$ $\mathcal{T}_{\tau} := \{0, 2\tau, \cdots, T\}, \quad \mathcal{T}_{\tau}' := \{\tau, 3\tau, \cdots, T - \tau\}.$

$$\partial_{\tau}u(t,x):=\frac{u(t+\tau,x)-u(t-\tau,x)}{2\tau}.$$

Spatial grid $\mathbb{T}_h:=\{0,h,\cdots,1-h\}$, where $1/h\!\in\!\mathbb{N}$.

▶ Unknowns $m: \mathcal{T}'_{\tau} \times \mathbb{T}_h \to \mathbb{R}$ and $\rho: \mathcal{T}_{\tau} \times \mathbb{T}_h \to \mathbb{R}$ are s.t.

$$\partial_{\tau}\rho = \mathsf{L}_{\mathsf{h}}\mathsf{m}, \qquad \qquad \rho(\mathsf{T},\cdot) = 1,$$

where L_h discretizes $-\operatorname{div}(D\nabla \cdot)$. For now assume $D=\operatorname{Id}$ and standard Laplacian discretization. $(e_i)_{i=1}^d$ can. basis

$$-L_h u(x) := \frac{1}{h^2} \sum_{i=1}^d [u(x + he_i) - 2u(x) + u(x - he_i)]$$

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The BBB formulation of evolution

Optimal transport

discretization

Harmonic time
averaging

Space discretization

Voronoi decomposition

equation

Easy path

Staggered time grids, with half timestep $\tau > 0$, $T/(2\tau) \in \mathbb{N}$

$$\mathcal{T}_{\tau}:=\{0,2\tau,\cdots,T\}, \quad \mathcal{T}_{\tau}':=\{\tau,3\tau,\cdots,T-\tau\}.$$

$$\partial_{\tau}u(t,x):=\frac{u(t+\tau,x)-u(t-\tau,x)}{2\tau}.$$

Spatial grid $\mathbb{T}_h := \{0, h, \cdots, 1-h\}$, where $1/h \in \mathbb{N}$. • Unknowns $m : \mathcal{T}'_{\tau} \times \mathbb{T}_h \to \mathbb{R}$ and $\rho : \mathcal{T}_{\tau} \times \mathbb{T}_h \to \mathbb{R}$ are s.t.

$$\partial_{\tau}\rho = L_h m, \qquad \qquad \rho(T, \cdot) = 1,$$

where L_h discretizes $-\operatorname{div}(D\nabla \cdot)$. For now assume $D=\operatorname{Id}$ and standard Laplacian discretization. $(e_i)_{i=1}^d$ can. basis

$$-L_h u(x) := \frac{1}{h^2} \sum_{i=1}^d [u(x + he_i) - 2u(x) + u(x - he_i)]$$

▶ Discretized BBB energy, with averaging operator A:

$$2\tau h^d \sum_{t \in \mathcal{T}_+', x \in \mathbb{T}_+^d} \left[\frac{m(t,x)^2}{2\mathcal{A}(\rho(t-\tau,x), \rho(t+\tau,x))} - m(t,x)u_0(x) \right].$$

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The BBB formulation

Optimal transpor

Optimal transport

discretization

Harmonic time averaging

discretization

Voronoi decomposition

accompositi

equatio

Easy path

Arithmetic average: $\mathcal{A}(\rho_-, \rho_+) = (1 - \theta)\rho_- + \theta\rho_+$.

$$\sum_{t \in \mathcal{T}_{\tau}', \ x \in \mathbb{T}_h^d} \left[\frac{m(t, x)^2}{2(1 - \theta)\rho(t - \tau, x) + 2\theta\rho(t + \tau, x)} - m(t, x)u_0(x) \right],$$
 where $\theta \in [0, 1]$.

Arithmetic mean $\theta = \frac{1}{2}$ typically used for OT discretization

In N. Papadakis, G. Peyré, E. Oudet, *Optimal transport* with proximal splitting. SIAM Imag Science (2014).

Discretization

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The BBB

Optimal transport Time

discretization Harmonic time averaging

Voronoi decomposition

Easy path selection

Arithmetic average: $\mathcal{A}(\rho_-, \rho_+) = (1 - \theta)\rho_- + \theta\rho_+$.

$$\sum_{t \in \mathcal{T}_{\tau}', \ x \in \mathbb{T}_h^d} \Big[\frac{m(t,x)^2}{2(1-\theta)\rho(t-\tau,x) + 2\theta\rho(t+\tau,x)} - m(t,x)u_0(x) \Big],$$
 where $\theta \in [0,1]$.

- Arithmetic mean $\theta = \frac{1}{2}$ typically used for OT discretization 🗓 N. Papadakis, G. Peyré, E. Oudet, Optimal transport with proximal splitting. SIAM Imag Science (2014).
 - ▶ A discrete duality argument shows that minimizing this energy is equivalent to solving the scheme

$$\partial_{\tau}u(t,\cdot)+rac{1}{2}\Big[\theta L_{h}u^{2}(t-\tau,\cdot)+(1-\theta)L_{h}u^{2}(t+\tau,\cdot)\Big]=0,$$

$$t \in \mathcal{T}'_{\tau}$$
, with initial condition $u(\tau) + \tau(1-\theta)L_hu^2(\tau) = u_0$.

- **Standard** θ -scheme for the QPME:
 - $\theta = 1$ Explicit scheme, first order accurate, with CFL.
- $\theta = \frac{1}{2}$ Semi-implicit scheme, second order accurate, with CFL. $\theta = 0$ Implicit scheme, first order accurate, without CFL.

formulation of PDEs lean-Marie Mirebeau

Discretization

of the BBB

The BBB

transport

Harmonic time

averaging

Easy path selection

 $\sum_{t\in\mathcal{T}_{\tau}',\,x\in\mathbb{T}_h^d}\left[\frac{m(t,x)^2}{4\rho(t-\tau,x)}+\frac{m(t,x)^2}{4\rho(t+\tau,x)}-m(t,x)u_0(x)\right].$

Harmonic average: $\mathcal{A}(\rho_-, \rho_+)^{-1} = \frac{1}{2}(\rho_-^{-1} + \rho_+^{-1})$

Optimality conditions similar to a mean field game

$$1 \qquad 4 \circ (+ -\pi)^2 > 1 \qquad 4 \circ$$

 $\partial_{\tau}u(t)+\frac{1}{4}L_h\left(u(t-\tau)^2\frac{A\rho(t-\tau)^2}{\rho(t)^2}\right)+\frac{1}{4}L_h\left(u(t+\tau)^2\frac{A\rho(t+\tau)^2}{\rho(t)^2}\right)=$

initial condition for
$$u$$
, and terminal condition $\rho(T) = 1$.

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Discretization

of the BBB formulation of PDEs

of evolution Optimal

transport Time

Harmonic time

Voronoi

Easy path selection

decomposition

discretization

averaging

Theorem (E. Stampfli, M, 2025)

Harmonic average: $\mathcal{A}(\rho_-, \rho_+)^{-1} = \frac{1}{2}(\rho_-^{-1} + \rho_+^{-1})$

Optimality conditions similar to a mean field game

Assume a smooth positive solution u of the QPME. Then

 $\sum_{t\in\mathcal{T}_{\tau}',\;x\in\mathbb{T}_{+}^{d}}\left[\frac{m(t,x)^{2}}{4\rho(t-\tau,x)}+\frac{m(t,x)^{2}}{4\rho(t+\tau,x)}-m(t,x)u_{0}(x)\right].$

initial condition for u, and terminal condition $\rho(T) = 1$.

 $\max_{t\in\mathcal{T}_{\tau}}\|\phi(t,\cdot)-\phi_h^{\tau}(t,\cdot)\|_{\ell^1(\mathbb{T}_h^d)}=\mathcal{O}(\tau^2+h^2).$

 ϕ, ϕ_h^{τ} are the continuous and discrete dual potentials. $\tau, h > 0$

Potential ϕ satisfies $m = \partial_{\tau} \phi$ and $\rho = 1 + L_h \phi$.

 $\partial_{\tau}u(t)+\frac{1}{4}L_h\left(u(t-\tau)^2\frac{A\rho(t-\tau)^2}{\rho(t)^2}\right)+\frac{1}{4}L_h\left(u(t+\tau)^2\frac{A\rho(t+\tau)^2}{\rho(t)^2}\right)=$

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The BBB of evolution

Harmonic time averaging

discretization Voronoi decomposition

Easy path selection

Discretization

of the BBB formulation

 $\sum_{t \in \mathcal{T}_{\tau}', \ x \in \mathbb{T}_{t}^{d}} \left[\frac{\partial_{\tau} \phi(t, x)^{2}}{1 + L_{h} \phi(t - \tau, x)} + \frac{\partial_{\tau} \phi(t, x)^{2}}{1 + L_{h} \phi(t + \tau, x)} - 4m(t, x)u_{0}(x) \right].$

$$t\in\mathcal{T}_{ au}',\,x\in\mathbb{T}_{h}^{d}$$

$$L_h(u(t-\tau))$$

initial condition for u, and terminal condition $\rho(T) = 1$.

$$\rho(t)^2$$

$$\rho(T) = 1$$

$$T)=1.$$

 ϕ, ϕ_h^{τ} are the continuous and discrete dual potentials. $\tau, h > 0$

 $\max_{t\in\mathcal{T}_{\tau}}\|\phi(t,\cdot)-\phi_h^{\tau}(t,\cdot)\|_{\ell^1(\mathbb{T}_h^d)}=\mathcal{O}(\tau^2+h^2).$

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Harmonic average: $\mathcal{A}(\rho_-, \rho_+)^{-1} = \frac{1}{2}(\rho_-^{-1} + \rho_+^{-1})$

- $\partial_{\tau} u(t) + \frac{1}{4} \mathcal{L}_h \left(u(t-\tau)^2 \frac{\mathcal{A}\rho(t-\tau)^2}{\rho(t)^2} \right) + \frac{1}{4} \mathcal{L}_h \left(u(t+\tau)^2 \frac{\mathcal{A}\rho(t+\tau)^2}{\rho(t)^2} \right) =$

- Optimality conditions similar to a mean field game

- Optimal
- transport
- Time

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The BBB formulation of evolution

Optimal

trans

discretizat

Harmonic time averaging

Space discretization

Voronoi decomposition

equation

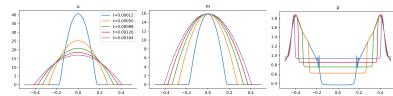
Easy path

Numerical experiment, using the Barenblatt profile.

Compactly supported, non-smooth solution of the QPME

$$u(t,x) := \frac{2}{t^{\alpha}} \max \left\{ 0, \gamma - \frac{\beta}{4} \frac{\|x\|^2}{t^{2\beta}} \right\},$$

$$\alpha:=rac{d}{d+2}, \beta:=rac{1}{d+2}.$$
 Obtained expressions of $m_T(t,x),
ho_T(t,x)$



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The BBB formulation of evolution

Optimal transport

Time

Harmonic time

Space discretization

Voronoi decomposition

Burgers equatior

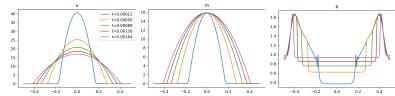
Easy path selection

Numerical experiment, using the Barenblatt profile.

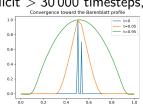
Compactly supported, non-smooth solution of the QPME

$$u(t,x) := \frac{2}{t^{\alpha}} \max \left\{ 0, \gamma - \frac{\beta}{4} \frac{\|x\|^2}{t^{2\beta}} \right\},$$

$$\alpha:=rac{d}{d+2}, \beta:=rac{1}{d+2}.$$
 Obtained expressions of $m_T(t,x),
ho_T(t,x)$



► Barenblatt profile is an attractor. BBB formulation with only 10 timesteps on [0,1]. (explicit > 30 000 timesteps, semi-implicit > 5 000 timesteps)



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The BBB formulatio

Optimal

transport

Time

Harmonic time

Space discretization

Voronoi decomposition

D.....

equation Easy path

selection

The BBB formulation of evolution PDEs Connection with optimal transport

Time discretization

Harmonic time averaging

Space discretization Voronoi/Selling decomposition

Burgers equation
Easy path selection

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The BBB formulatio of evolution

Optimal

Time

discretization

Harmonic time averaging

Space discretization

Voronoi decomposition

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equation Easy path

Multi-dimensional anisotropic QPME

▶ Consider a stencil $E \subseteq \mathbb{Z}^d$ and smooth weights λ^e s.t.

$$\mathcal{D}(x) = \sum_{e \in E} \lambda^e(x) e e^{\top}$$

▶ Monotone numerical scheme $-L_h u = \text{div}(\mathcal{D}\nabla u) + \mathcal{O}(h^2)$,

$$-L_h u(x) = \sum_{\substack{e \in E \\ \nu = +}} \lambda^e(x + \frac{1}{2}h\nu e) \frac{u(x + h\nu e) - u(x)}{h^2}$$

► Same convergence result and numerical approach (primal-dual algorithm using a space-time FFT).

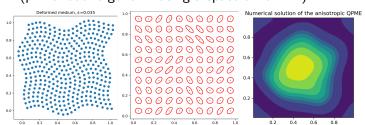


Figure: Synthetic two dimensional experiment (deformed medium)

Discretization

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The BBB formulation of evolution

Optimal transport

Time

discretization

Harmonic time averaging

discretization

Voronoi decomposition

equation

Easy path

Voronoi/Selling decomp. of positive quadratic forms

We use adaptive finite difference based on the decomposition of the anisotropy matrix $D \in \mathcal{S}_d^{++}$ with non-negative weights $\lambda^e \geq 0$, integer offsets $e \in \mathbb{Z}^d$.

$$\sum_{\mathsf{e}\in\mathbb{Z}^d}\lambda^\mathsf{e}\mathsf{e}\mathsf{e}^\top=D$$

F. Bonnans, G. Bonnet, M, Monotone Discretization of Anisotropic Differential Operators Using Voronoi's First Reduction. Constr. Approx. (2023)

Discretization

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The BBB formulation

PDEs Optimal transport

Time

discretizatio

Harmonic time averaging

discretization

Voronoi decomposition

Burgers

Easy path

Voronoi/Selling decomp. of positive quadratic forms

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- ► Voronoi/Selling selection principle:

$$\max_{\lambda:\mathbb{Z}^d\to [0,\infty[}\sum_{e\in\mathbb{Z}^d}\lambda^e\quad\text{subject to }\sum_{e\in\mathbb{Z}^d}\lambda^eee^\top=D.$$

(There is a solution with $\leq d(d+1)/2$ positive coefficients.)

F. Bonnans, G. Bonnet, M, Monotone Discretization of Anisotropic Differential Operators Using Voronoi's First Reduction. Constr. Approx. (2023)

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Discretization

The BBB

Optimal

transport

Time

Harmonic time averaging

Voronoi decomposition

Easy path selection

Voronoi/Selling decomp. of positive quadratic forms

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- (There is a solution with $\leq d(d+1)/2$ positive coefficients.)
- Dual linear program:

$$\min_{M \in \mathcal{S}_d} \mathsf{Tr}(\mathit{DM}) \quad \text{s.t. } \forall e \in \mathbb{Z}^d \setminus \{0\}, \ \|e\|_M^2 := \langle e, \mathit{Me} \rangle \geq 1.$$

- Periodic sphere packing pb: replace Tr(DM) with det(M).
- F. Bonnans, G. Bonnet, M. Monotone Discretization of Anisotropic Differential Operators Using Voronoi's First Reduction. Constr. Approx. (2023)

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Optimal transport

Time

discretization

Harmonic time averaging

Space discretization

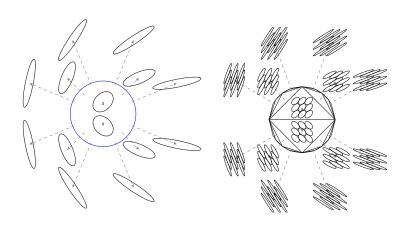
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Voronoi decomposition

Burgers

Easy path

- ▶ Left: Unit ball defined by $D = \begin{pmatrix} 1+a & b \\ b & 1-a \end{pmatrix}$, $a^2 + b^2 < 1$.
- ▶ Right: Linear program minimizer 2*M*. Support of decomp.



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Optimal transport

Lim

discretization

Harmonic time averaging

Space

discretization

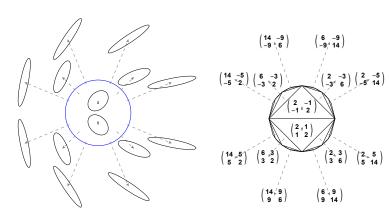
Voronoi decomposition

Burgers

equation

Easy path selection

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The BBB formulation of evolution

Optimal transport

Time

discretization

Harmonic time averaging

Space

discretization

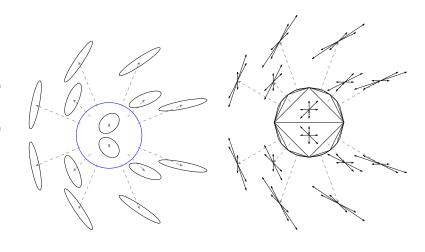
Voronoi decomposition

Burgers

equatio

Easy path selection

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The BBB

Optimal

transport

Harmonic time averaging

discretization Voronoi

decomposition

Easy path selection

Theorem (Properties of Voronoi's decomposition)

There are computable coefficients $\lambda^e \in \text{Lip}(\mathcal{S}_d^{++}, [0, \infty[) \text{ s.t.})$

- \blacktriangleright (Consistency) $D = \sum_{e \in \mathbb{Z}^d} \lambda^e(D) e e^{\top}$
- (Support) cardinality: $\#\{e \in \mathbb{Z}^d \mid \lambda^e(D) \neq 0\} \leq N(d)$, and intrinsic radius: $\lambda^{e}(D) \neq 0 \Rightarrow \|e\|_{D^{-1}} < R(d)\|D^{-\frac{1}{2}}\|$ \triangleright (Spanning, if d < 4) $\exists e_1, \dots, e_d \in \mathbb{Z}^d$.
- $|\det(e_1,\cdots,e_d)|=1$ and $\lambda^{e_1}(D),\cdots,\lambda^{e_d}(D)>0$.
- (Unimodular inv.) $\lambda^e(D) = \lambda^{Ae}(ADA^\top), \forall A \in GL(\mathbb{Z}^d).$

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The BBB

Optimal

Voronoi decomposition

transport

of evolution

Harmonic time averaging

Discretization of the BBB

Theorem (Properties of Voronoi's decomposition) There are computable coefficients $\lambda^e \in C^{\infty}(\mathcal{S}_d^{++}, [0, \infty[) \text{ s.t.})$

(Consistency)
$$D = \sum_{e \in \mathbb{Z}^d} \lambda(D) ee$$

• (Support) cardinality: $\#\{e \in \mathbb{Z}^d \mid \lambda^e(D) \neq 0\} \leq N(d)$,

anisotropic wave equation, preprint, 2025

and intrinsic radius: $\lambda^e(D) \neq 0 \Rightarrow \|e\|_{D^{-1}} \leq R(d)\|D^{-\frac{1}{2}}\|$ $(Spanning, if d < 6) \exists e_1, \dots, e_d \in \mathbb{Z}^d.$

 $|\det(e_1,\cdots,e_d)|=1$ and $\lambda^{e_1}(D),\cdots,\lambda^{e_d}(D)>0$.

► (Unimodular inv.) $\lambda^e(D) = \lambda^{Ae}(ADA^\top)$, $\forall A \in GL(\mathbb{Z}^d)$. Variant with smooth coefficients, obtained by considering a

Variant with smooth coefficients, obtained by considering a smooth strictly convex variant of Voronoi's linear program.

Simple the strictly convex variant of voronors linear progra

$$\max_{\lambda} \sum_{e \in \text{supp}(\rho)} \lambda^e - \delta \rho^e \mathcal{B}(\frac{\lambda^e}{\rho^e}) \qquad \text{subject to } D = \sum_{e \in \text{supp}(\rho)} \lambda^e e e^\top$$

Barrier fct $\mathcal{B}(s) := \frac{1}{2}s^2 - \ln s$. Carefully chosen weights $\rho_e(D)$.

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equation Easy path selection

Mirebeau

Voronoi decomposition

Easy path

Structure preserving anisotropic PDEs on grids

Based on Voronoi's decomp: $D = \sum_{e \in E} \lambda^e e e^{\top}, \ \lambda^e \ge 0, \ E \subseteq \mathbb{Z}^d$

▶ Causal schemes for eikonal type PDEs, $\|v\|_D := \sqrt{v^\top Dv}$

The BBB formulation of evolution PDEs
$$\|\nabla u(x)\|_D^2 = \sum_{e \in E} \frac{\lambda^e}{h^2} \max\{0, u(x) - u(x - he), u(x) - u(x + he)\}^2 + \mathcal{O}(h)$$

of evolution " h^2 h^2 h^2 Optimal transport

Harmonic time
averaging

Space
discretization

Guillaume Bonnet, M, Monotone discretization of the Monge-Ampère equation of optimal transport, M2AN, 2022

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Discretization

of the BBB

Structure preserving anisotropic PDEs on grids Based on Voronoi's decomp: $D = \sum_{e \in F} \lambda^e e e^{\top}, \lambda^e \ge 0, E \subseteq \mathbb{Z}^d$

Causal schemes for eikonal type PDEs,
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The BBB formulation of evolution PDEs optimal
$$\|\nabla u(x)\|_D^2 = \sum_{e \in E} \frac{\lambda^e}{h^2} \max\{0, u(x) - u(x - he), u(x) - u(x + he)\}^2 + \mathcal{O}(h)$$

Monotone schemes for degenerate elliptic PDEs

$$\operatorname{Tr}(D\nabla^2 u(x)) = \sum_{e \in E} \lambda^e \frac{u(x+he) - 2u(x) + u(x-he)}{h^2} + \mathcal{O}(h^2)$$

Harmonic time

transport

Easy path

selection

🖺 M. Riemannian Fast-Marching on Cartesian Grids, Using Voronoi's First Reduction of Quadratic Forms, SINUM, 2019

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Discretization

of the BBB formulation

Structure preserving anisotropic PDEs on grids Based on Voronoi's decomp: $D = \sum_{e \in F} \lambda^e e e^{\top}, \ \lambda^e \ge 0, \ E \subseteq \mathbb{Z}^d$ ► Causal schemes for eikonal type PDEs, $||v||_D := \sqrt{v^T Dv}$

The BBB

 $\|\nabla u(x)\|_{D}^{2} = \sum_{e \in F} \frac{\lambda^{e}}{h^{2}} \max\{0, u(x) - u(x - he), u(x) - u(x + he)\}^{2} + \mathcal{O}(h)$

Optimal transport

► Monotone schemes for degenerate elliptic PDEs

anisotropic wave equation, preprint, 2025

Harmonic time averaging

 $\operatorname{Tr}(D\nabla^2 u(x)) = \sum \lambda^e \frac{u(x+he) - 2u(x) + u(x-he)}{h^2} + \mathcal{O}(h^2)$

Voronoi decomposition

▶ Low dispersion error scheme for wave eq. $\partial_{tt}q = \text{div}(D\nabla q)$

Easy path

selection

 $\frac{q_{n+1}(x) - 2q_n(x) + q_{n-1}(x)}{\tau^2} = \sum_{e \in E} \lambda^e(x + \frac{1}{2}h\nu e) \frac{q_n(x + h\nu e) - q_n(x)}{h^2}$

M. Haloui, L. Métivier, M, Selling's decomposition and the

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The BBB formulation

Optimal

transport

discretization

Harmonic time averaging

discretization

Voronoi decomposition

Burgers equation

Easy path

The BBB formulation of evolution PDEs Connection with optimal transport

Time discretization

Harmonic time averaging

Space discretization
Voronoi/Selling decomposition

Burgers equation
Easy path selection

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The BBB formulation of evolution

Optimal transport

Time

discretization

Harmonic time averaging

Space

discretization

Voronoi decomposition

Burgers equation

Easy path

Burgers equation : $\partial_t u + \frac{1}{2} \partial_x u^2 = \nu \partial_{xx} u$, $\nu \geq 0$.

BBB formulation: minimize kinetic energy among weak solutions

$$\inf_{u(0)=u_0} \sup_{\phi(T)=0} \int_{[0,T]\times \mathbb{T}} \frac{1}{2} u^2 + \underbrace{\left(\partial_t u + \frac{1}{2} \partial_x u^2 - \nu \partial_{xx} u\right) \phi}_{\text{Understood in the weak sense}}$$

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The BBB formulation of evolution

Optimal transport

Tin

discretizat

Harmonic time averaging

Space

Voronoi

decomposition

Burgers equation

Easy path

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$$\stackrel{?}{=} \sup_{\phi(T)=0} \inf_{u(0)=u_0} \int_{[0,T]\times\mathbb{T}} \frac{1}{2} u^2 (1-\partial_x \phi) - u(\partial_t \phi + \nu \partial_{xx} \phi) + u_0 \partial_t \phi$$

Under the assumption $\partial_x \phi < 1$, and with the relations

$$(1 - \partial_{\mathsf{x}}\phi)u = \partial_{\mathsf{t}}\phi + \nu\partial_{\mathsf{x}\mathsf{x}}\phi, \qquad \phi(T) = 0$$

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The BBB formulation of evolution

Optimal

Harmonic time averaging

Space

discretization Voronoi

decomposition

Burgers equation

Easy path selection

Burgers equation : $\partial_t u + \frac{1}{2} \partial_x u^2 = \nu \partial_{xx} u$, $\nu \geq 0$.

BBB formulation: minimize kinetic energy among weak solutions

$$\begin{split} &\inf_{u(0)=u_0}\sup_{\phi(T)=0}\int_{[0,T]\times\mathbb{T}}\frac{1}{2}u^2+\underbrace{(\partial_t u+\frac{1}{2}\partial_x u^2-\nu\partial_{xx}u)\phi}_{\text{Understood in the weak sense}}\\ &\stackrel{?}{=}\sup_{\phi(T)=0}\inf_{u(0)=u_0}\int_{[0,T]\times\mathbb{T}}\frac{1}{2}u^2(1-\partial_x\phi)-u(\partial_t\phi+\nu\partial_{xx}\phi)+u_0\partial_t\phi\\ &=-\inf_{\phi(T)=0}\int_{[0,T]\times\mathbb{T}}\frac{(\partial_t\phi+\nu\partial_{xx}\phi)^2}{2(1-\partial_x\phi)}-u_0\partial_t\phi \end{split}$$

Under the assumption $\partial_x \phi < 1$, and with the relations

$$(1 - \partial_{\mathsf{x}}\phi)u = \partial_{\mathsf{t}}\phi + \nu\partial_{\mathsf{x}\mathsf{x}}\phi, \qquad \phi(T) = 0$$

Jean-Marie Mirebeau

The BBB of evolution

Optimal transport

Harmonic time averaging

Voronoi

decomposition

Burgers equation

Easy path selection

Burgers equation : $\partial_t u + \frac{1}{2} \partial_x u^2 = \nu \partial_{xx} u$, $\nu \ge 0$. BBB formulation: minimize kinetic energy among weak solutions

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Understood in the weak sense
$$\stackrel{?}{=} \sup_{\phi(T)=0} \inf_{u(0)=u_0} \int_{[0,T]\times\mathbb{T}} \frac{1}{2} u^2 (1-\partial_x \phi) - u(\partial_t \phi + \nu \partial_{xx} \phi) + u_0 \partial_t \phi$$

$$= -\inf_{\phi(T)=0} \int_{[0,T]\times\mathbb{T}} \frac{(\partial_t \phi + \nu \partial_{xx} \phi)^2}{2(1 - \partial_x \phi)} - u_0 \partial_t \phi$$

$$= -\inf_{m,\rho} \int_{[0,T]\times\mathbb{T}} \frac{(m - \nu \partial_x \rho)^2}{2\rho} - u_0 m, \text{ s.t. } \partial_t \rho + \partial_x m = 0, \rho(T) = 1.$$

Under the assumption $\partial_x \phi < 1$, and with the relations

$$\rho = 1 - \partial_{\mathsf{x}} \phi, \qquad \mathsf{m} = \partial_{\mathsf{t}} \phi, \qquad \rho \mathsf{u} = \mathsf{m} - \nu \partial_{\mathsf{x}} \rho.$$

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Discretization of the BBB

> $\inf_{\rho,m} \int_{\mathbb{I}_0} \frac{(m - \nu \partial_x \rho)^2}{2\rho} - u_0 m, \text{ s.t. } \partial_t \rho + \partial_x m = 0, \rho(T) = 1.$ \triangleright Discretized BBB energy, half timestep τ , half gridscale h $\sum_{\substack{t \in \mathcal{T}_{\tau}' \\ x \in \mathbb{T}_{h}}} \left(\frac{1}{4} \sum_{\substack{\sigma_{t} = \pm \\ \sigma_{x} = \pm}} \frac{\left(m(t, x) - \nu \partial_{h} \rho(t + \sigma_{t} \tau, x) \right)^{2}}{2\rho(t + \sigma_{t} \tau, x + \sigma_{x} h)} - m(t, x) u_{0}(x) \right),$

▶ BBB formulation of Burgers' equation, $\nu \geq 0$

discretization

Voronoi decomposition ▶ Staggered time and space grids $m \in \mathbb{R}^{\mathcal{T}'_{\tau} \times \mathbb{T}_h}$, $\rho \in \mathbb{R}^{\mathcal{T}_{\tau} \times \mathbb{T}'_h}$.

subject to
$$\partial_{\tau}\rho + \partial_{h}m = 0$$
, and $\rho(T) = 1$.

e grids
$$m \in \mathbb{R}^{\mathcal{T}_{ au}' imes \mathbb{T}_h}$$
, $ho \in \mathbb{R}^{\mathcal{T}_{ au} imes \mathbb{T}_h'}$.

Assume a smooth positive solution on [0, T], with $\nu \geq 0$. Then

$$\max_{t \in \mathcal{T}_{ au}} \|\phi(t,\cdot) - \phi_{ au h}(t,\cdot)\|_{\ell^1(\mathbb{T}^d_h)} = \mathcal{O}(au^2 + h^2),$$

 $\phi, \phi_{\tau h}$ are the continuous and discrete dual potentials. $\tau, h > 0$

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The BBB formulation of evolution

Optimal transport

Time

discretization

Harmonic time averaging

Space discretization

Voronoi decomposition

Burgers equation

Easy path

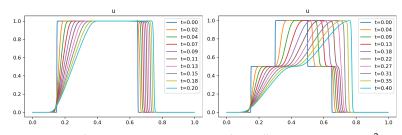


Figure: Solving Burgers equation with small viscosity, $\nu=10^{-3}.$

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Optimal transport

Time

Harmonic time averaging

discretization

Voronoi decomposition

Burgers

equation Easy path

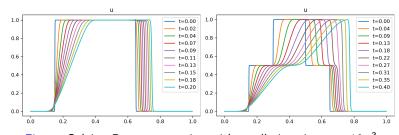


Figure: Solving Burgers equation with small viscosity, $\nu=10^{-3}$.

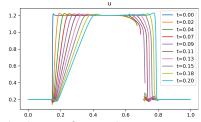


Figure: Numerical solution of inviscid Burgers, $\nu = 0$. The final time is correct, but the intermediate times are not reconstructed!

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Optimal transport

Time

dicaretiza

Harmonic time averaging

discretization

Voronoi decomposition

Burgers

Easy path

Brenier/Gallouët mountain climbing analogy



Figure: Mt. Everest to Lhotse along the crest is still an open problem.

- ▶ The BBB formulation of Burger's equation selects the weak solution with the correct final value and the fewest shocks.
- ▶ PDE \approx path along crest. BBB \approx path through valley.

Y. Brenier, The initial value problem for the Euler equations of incompressible fluids viewed as a concave maximization problem. Comm Math. Physics (2018).

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of evolut PDEs

Optimal transport

Time

Harmonic time

averaging

discretization

Voronoi decomposition

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Easy path

Conclusion:

- BBB formulation turns evolution PDEs into global optimization problems with nice convexity properties.
- Proximal primal-dual algo, using space-time FFT. No CFL.
- Selling-based Laplacian discretization has many other uses.
- ▶ Unconventional approach: solve only the final time $u(T, \cdot)$!

Perspectives:

- Fluid mechanics PDEs.
- ▶ Weighted kinetic energy $\int_{[0,T]\times\mathbb{T}^d} e^{-\gamma t} \|u(x,t)\|^2 dx dt$.

Happy birthday and thank you Albert!

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The BBB formulation of evolution

Optimal transport

Tim

discretizațio

Harmonic time averaging

Space

discretization Voronoi

decomposition

Burgers

equatio

Easy path selection

$$\operatorname{Tr}(D\nabla^2 u(x)) = \sum_{e \in E} \lambda^e \frac{u(x+he) - 2u(x) + u(x-he)}{h^2} + \mathcal{O}(h^2)$$

Monotone discretization of Monge-Ampere, via

$$d\det(\nabla^2 u)^{\frac{1}{d}}=\inf\{\operatorname{Tr}(D\nabla^2 u)\mid D\in\mathcal{S}_d^{++},\det D=1\}$$

Guillaume Bonnet, M, Monotone discretization of the Monge-Ampère equation of optimal transport, M2AN, 2022

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Optimal transport

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Harmonic time

Space discretization

Voronoi decomposition

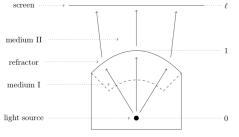
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equation Easy path selection

$\operatorname{Tr}(D\nabla^2 u(x)) = \sum_{e \in E} \lambda^e \frac{u(x+he) - 2u(x) + u(x-he)}{h^2} + \mathcal{O}(h^2)$

► Monotone discretization of Monge-Ampere, via

$$d\det(\nabla^2 u)^{\frac{1}{d}} = \inf\{\operatorname{Tr}(D\nabla^2 u) \mid D \in \mathcal{S}_d^{++}, \det D = 1\}$$



Designing a refractor projecting a given image amounts to solve

$$\det\left(\nabla^2 u(x) - A(x, \nabla u(x))\right) = B(x, \nabla u(x)),$$

$$x \in \Omega$$
, with boundary conditions $\nabla u(x) \in P(x)$, $x \in \partial \Omega$.

Guillaume Bonnet, M, Monotone discretization of the Monge-Ampère equation of optimal transport, M2AN, 2022

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Optimal transport

Time

Harmonic time averaging

Space discretizatio

Voronoi decomposition

Burgers

Easy path



Figure: Left: image to reproduce. Right: Appelseed® render.

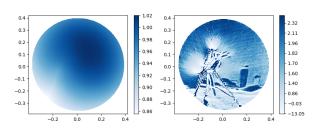


Figure: Left: refractor. Right: curvature of refractor.

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Optimal transport

Time

discretization

Harmonic time averaging

Space discretization

Voronoi decomposition

D....

equation

Easy path selection $\|\nabla u\|_D^2 = \sum_{e \in F} \frac{\lambda^e}{h^2} \max\{0, u(x) - u(x - he), u(x) - u(x + he)\}^2 + \mathcal{O}(h)$

- lacktriangle Causal scheme (FMM solvable) for eikonal PDE $\|
 abla u\|_D=1$
- Applications to path planning and tubular segmentation.

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The BBB

formulation

Optimal

Time

discretization

Harmonic time
averaging

Space discretization

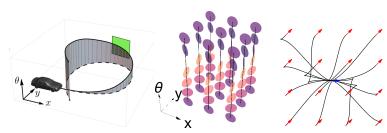
Voronoi

decomposition

Easy path

$\|\nabla u\|_D^2 = \sum_{e \in F} \frac{\lambda^e}{h^2} \max\{0, u(x) - u(x - he), u(x) - u(x + he)\}^2 + \mathcal{O}(h)$

- ightharpoonup Causal scheme (FMM solvable) for eikonal PDE $\|
 abla u\|_D=1$
- ▶ Applications to path planning and tubular segmentation.
- Ex: Reeds-Shepp sub-Riemannian vehicle model.



Position-orientation state space $\mathbb{M}:=\mathbb{R}^2_{x}\times\mathbb{S}^1_{\theta}$, anisotropic eikonal equation with relaxation parameter $\varepsilon>0$

$$\langle \nabla_{\mathbf{x}} u, \mathbf{n}(\theta) \rangle^2 + \varepsilon^2 \langle \nabla_{\mathbf{x}} u, \mathbf{n}(\theta)^{\perp} \rangle^2 + (\partial_{\theta} u)^2 = c(\mathbf{x}, \theta)^2,$$

where $n(\theta) = (\cos \theta, \sin \theta)$, and $c(x, \theta)$ is a cost function.

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The BBB formulatio of evolution PDEs

Optimal transport

Time

Harmonic time

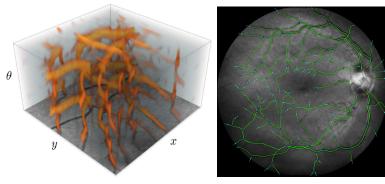
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discretization

Voronoi decomposition

Burgers

Easy path selection



Left: cost function $c(x, y, \theta)$ processed from a retina scan. Right: Reeds-Shepp vehicle minimal paths.

☐ G. Sanguinetti, E. Bekkers, R. Duits, M. Jansen, M. Mashtakov, M, *Sub-Riemannian fast marching in* SE(2), Iberoamrican congress on Pattern recognition, 2015

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Optimal transport

Voronoi decomposition

Easy path selection

Harmonic time averaging

Discretization

▶ Discretizes the wave equation $\partial_{tt}q = \text{div}(\mathcal{D}\nabla q)$.

 $\frac{q_{n+1}(x) - 2q_n(x) + q_{n-1}(x)}{2} = \sum_{n=1}^{\infty} \lambda^e(x + \frac{1}{2}h\nu e) \frac{q_n(x + h\nu e) - q_n(x)}{h^2}$

Guarantees against checkerboard artifacts (Spanning prop).

Fourth order variant. Cy rates if smooth coefficients λ^e .

M. Haloui, L. Métivier, M, Selling's decomposition and the anisotropic wave equation, preprint, 2025

formulation of PDEs lean-Marie Mirebeau

Discretization

of the BBB

 $\frac{q_{n+1}(x) - 2q_n(x) + q_{n-1}(x)}{\tau^2} = \sum \lambda^e(x + \frac{1}{2}h\nu e) \frac{q_n(x + h\nu e) - q_n(x)}{h^2}$

Guarantees against checkerboard artifacts (Spanning prop).

• Fourth order variant. Cy rates if smooth coefficients λ^e .

Theorem (Reduced dispersion error of the Selling scheme)

Consider the Fourier symbol assoc. to the Selling based scheme

 $\beta_{\mathit{h}}(\xi) := \sum \lambda^{\mathit{e}} \operatorname{sinc}(\tfrac{\mathit{h}}{2} \langle \xi, \mathit{e} \rangle)^{2}, \quad \textit{ where } D = \sum \lambda^{\mathit{e}} \mathit{ee}^{\top}.$

 $|\beta_h(\xi) - \|\xi\|_D^2 | < C(d) h^2 \|\xi\|_D^4 \|D^{-1}\|.$

Not satisfied by axis-aligned and criss-cross schemes, any C(d). 🖺 M. Haloui, L. Métivier, M, Selling's decomposition and the

▶ Discretizes the wave equation $\partial_{tt}q = \text{div}(\mathcal{D}\nabla q)$.

Then for all $\xi \in \mathbb{R}^d$ and $D \in \mathcal{S}_d^{++}$,

anisotropic wave equation, preprint, 2025

The BBB

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Voronoi decomposition

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The BBB formulation of evolution

Optimal transport

Time

discretization

Harmonic time averaging

discretization

Voronoi decomposition

Burgers

Easy path

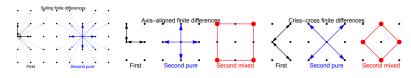


Figure: The Selling scheme avoids four-point mixed finite differences.

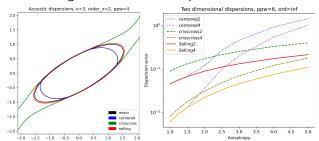
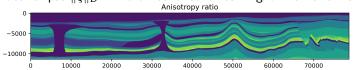


Figure: The Selling-scheme dispersion curve $\beta_h(\xi) = 1$ is closer to the ideal ellipse $\|\xi\|_D = 1$ than other schemes. Right: max error vs aniso.





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Optimal

Time

discretizatio

averaging

discretization

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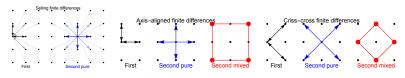


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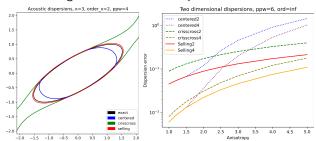
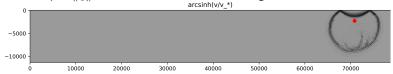


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The BBB formulation of evolution

Optimal

Time

discretization

Harmonic time averaging

discretization

decomposition

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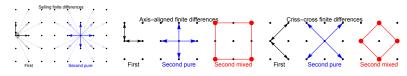


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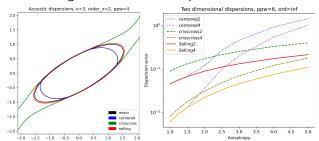
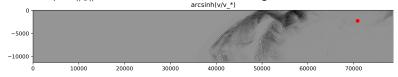


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The BBB formulation

Optimal transport

Time

discretizatio

Harmonic time averaging

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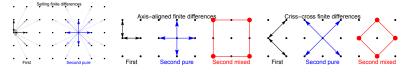


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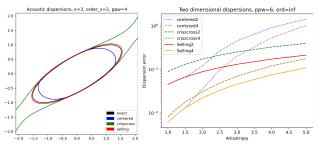


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