Discretization of the Ballistic-Benamou-Brenier formulation of the porous medium equation

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Nonlinear approximation for High-Dimensional problems
Workshop in honor of Albert Cohen
In collaboration with E. Stampfli, Y. Brenier and T. Gallouet.

Erwan Stampfli, M, *Discretization and convergence of the ballistic Benamou-Brenier formulation of the porous medium and Burgers' equations*, preprint, 2025

The quadratic porous medium equation (QPME) reads

$$\partial_t u = \frac{1}{2}\operatorname{div}(\mathcal{D}\nabla u^2), \qquad u(t=0) = u_0.$$

Solved over the time interval $[0, T]$, domain $\mathbb{T}^d = (\mathbb{R}/\mathbb{Z})^d$, initial data $u_0 \geq 0$, smooth diffusion tensors $\mathcal{D} : \mathbb{T}^d \to \mathcal{S}_d^{++}$.

- Reformulation by Y. Brenier as a convex optimization problem in space and time, related with optimal transport.
- Unconventional numerical method, relying on space-time FFT and proximal operators. No CFL, 2nd order.
- Applies to various conservation equations, fluid mechanics.

Y. Brenier, *Examples of hidden convexity in nonlinear PDEs.* Book available online (2020).

D. Vorotnikov, *Hidden convexity and Dafermos' principle for some dispersive equations.* arXiv (2025)
"The physical solution dissipates entropy earliest and fastest"

S. Singh, J. Ginster, A. Acharya, *A hidden convexity of nonlinear elasticity.* Journal of Elasticity (2024).

- Non-linear PDEs often do not admit smooth solutions.
- Consider the weak form: $\forall \phi \in C_c^\infty([0, T[\times \mathbb{T}^d)$

$$\int_{[0,T]\times\mathbb{T}^d} (\partial_t u - \tfrac{1}{2}\operatorname{div}(\mathcal{D}\nabla u^2))\phi = \int_{[0,T]\times\mathbb{T}^d} -(u-u_0)\partial_t\phi - \tfrac{1}{2}u^2\operatorname{div}(\mathcal{D}\nabla\phi)$$

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- Ex: Barenblatt profile (non-smooth, compact support)
- Issue: possibly several weak solutions.
  (Are there non-positive QPME solutions ?)

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BBB selection principle: minimize the total kinetic energy $\frac{1}{2}\int_{[0,T]\times\mathbb{T}^d} u^2$ (or a suitable entropy) among all weak solutions

$$\inf_{u(0)=u_0} \sup_{\phi(T)=0} \int_{[0,T]\times\mathbb{T}^d} \tfrac{1}{2} u^2 + \underbrace{(\partial_t u - \tfrac{1}{2}\operatorname{div}(\mathcal{D}\nabla u^2))\phi}_{\text{Understood in weak sense}}$$

Project the null function onto the manifold of weak solutions.
(Acharya et al often project an initial guess u_{ref})

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$$\overset{?}{=} \sup_{\phi(T)=0} \inf_{u(0)=u_0} \int_{[0,T]\times\mathbb{T}^d} \frac{1}{2}u^2(1 - \operatorname{div}(\mathcal{D}\nabla\phi)) - (u-u_0)\partial_t\phi.$$

- Question mark: does duality hold ? (We'll come back to this)

Dual formulation

- Assuming duality holds, kinetic energy minimization reads

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- Minimize pointwise w.r.t. u, assuming $\operatorname{div}(\mathcal{D}\nabla\phi) < 1$:

$$u = \frac{\partial_t\phi}{1-\operatorname{div}(\mathcal{D}\nabla\phi)}$$

Convex optimization problem w.r.t. ϕ

$$\inf_{\phi(T)=0} \int_{[0,T]\times\mathbb{T}^d} \frac{(\partial_t\phi)^2}{2(1-\operatorname{div}(\mathcal{D}\nabla\phi))} - u_0\partial_t\phi.$$

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Ratio understood as the convex l.s.c. perspective function

$$\mathcal{P}(a,b) := \begin{cases} a^2/(2b) & \text{if } b>0, \\ 0 & \text{if } a=b=0, \\ \infty & \text{else.} \end{cases}$$

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- Letting $m := \partial_t\phi$ and $\rho := 1-\mathrm{div}(\mathcal{D}\nabla\phi)$ we find

$$\inf_{m,\rho} \int_{[0,T]\times\mathbb{T}^d} \frac{m^2}{2\rho} - u_0 m, \quad \text{s.t. } \partial_t\rho + \mathrm{div}(\mathcal{D}\nabla m) = 0,\ \rho(T) = 1.$$

A ballistic variant of optimal transport

The Benamou-Brenier formulation of optimal transport reads

$$\inf_{m,\rho} \int_{[0,1]\times\Omega} \frac{|m|^2}{2\rho}, \quad \text{s.t. } \partial_t \rho + \operatorname{div} m = 0,\ \rho(0) = \rho_0, \rho(1) = \rho_1,$$

where ρ_0, ρ_1 are probability densities on Ω.

Similar structure as the QPME reformulation, with the caveats:

- Boundary conditions at both endpoints, vs $\rho(T) = 1$.
- First order continuity equation, vs $\partial_t \rho + \operatorname{div}(\mathcal{D}\nabla m) = 0$.
- Absence of the first order term, vs $-u_0 m$.

J.-D. Benamou, Y. Brenier, *A computational fluid mechanics solution to the Monge-Kantorovich mass transfer problem*. Numer Math (2000).

- Staggered time grids, with half timestep $\tau > 0$, $T/(2\tau) \in \mathbb{N}$

$$\mathcal{T}_\tau := \{0, 2\tau, \cdots, T\}, \qquad \mathcal{T}'_\tau := \{\tau, 3\tau, \cdots, T-\tau\}.$$

$$\partial_\tau u(t,x) := \frac{u(t+\tau,x) - u(t-\tau,x)}{2\tau}.$$

Spatial grid $\mathbb{T}_h := \{0, h, \cdots, 1-h\}$, where $1/h \in \mathbb{N}$.

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- Unknowns $m : \mathcal{T}'_\tau \times \mathbb{T}_h \to \mathbb{R}$ and $\rho : \mathcal{T}_\tau \times \mathbb{T}_h \to \mathbb{R}$ are s.t.

$$\partial_\tau \rho = L_h m, \qquad \rho(T, \cdot) = 1,$$

where L_h discretizes $-\operatorname{div}(D\nabla\cdot)$. For now assume $D = \mathrm{Id}$ and standard Laplacian discretization. $(e_i)_{i=1}^d$ can. basis

$$-L_h u(x) := \frac{1}{h^2}\sum_{i=1}^d [u(x+he_i) - 2u(x) + u(x-he_i)]$$

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- Discretized BBB energy, with averaging operator $\mathcal{A}$:

$$2\tau h^d \sum_{t\in\mathcal{T}'_\tau,\, x\in\mathbb{T}_h^d} \Big[\frac{m(t,x)^2}{2\mathcal{A}\big(\rho(t-\tau,x), \rho(t+\tau,x)\big)} - m(t,x) u_0(x)\Big].$$

Arithmetic average: $\mathcal{A}(\rho_-, \rho_+) = (1-\theta)\rho_- + \theta\rho_+$.

$$\sum_{t \in \mathcal{T}'_\tau,\, x \in \mathbb{T}_h^d} \Big[\frac{m(t,x)^2}{2(1-\theta)\rho(t-\tau,x) + 2\theta\rho(t+\tau,x)} - m(t,x)u_0(x)\Big],$$

where $\theta \in [0,1]$.

- Arithmetic mean $\theta = \frac{1}{2}$ typically used for OT discretization
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- A discrete duality argument shows that minimizing this energy is equivalent to solving the scheme

$$\partial_\tau u(t,\cdot) + \frac{1}{2}\Big[\theta L_h u^2(t-\tau,\cdot) + (1-\theta) L_h u^2(t+\tau,\cdot)\Big] = 0,$$

$t \in \mathcal{T}'_\tau$, with initial condition $u(\tau) + \tau(1-\theta)L_h u^2(\tau) = u_0$.

- Standard θ-scheme for the QPME:

$\theta = 1$ Explicit scheme, first order accurate, with CFL.
$\theta = \frac{1}{2}$ Semi-implicit scheme, second order accurate, with CFL.
$\theta = 0$ Implicit scheme, first order accurate, without CFL.

Harmonic average: $\mathcal{A}(\rho_-, \rho_+)^{-1} = \frac{1}{2}(\rho_-^{-1} + \rho_+^{-1})$

$$\sum_{t\in\mathcal{T}_\tau',\, x\in\mathbb{T}_h^d} \Big[\frac{m(t,x)^2}{4\rho(t-\tau,x)} + \frac{m(t,x)^2}{4\rho(t+\tau,x)} - m(t,x)u_0(x)\Big].$$

- Optimality conditions similar to a mean field game

$$\partial_\tau u(t)+\frac{1}{4}L_h\Big(u(t-\tau)^2\,\frac{\mathcal{A}\rho(t-\tau)^2}{\rho(t)^2}\Big)+\frac{1}{4}L_h\Big(u(t+\tau)^2\,\frac{\mathcal{A}\rho(t+\tau)^2}{\rho(t)^2}\Big) =$$

initial condition for u, and terminal condition $\rho(T) = 1$.

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Theorem (E. Stampfli, M, 2025)

Assume a smooth positive solution u of the QPME. Then

$$\max_{t \in \mathcal{T}_\tau} \|\phi(t,\cdot) - \phi_h^\tau(t,\cdot)\|_{\ell^1(\mathbb{T}_h^d)} = \mathcal{O}(\tau^2 + h^2).$$

ϕ, ϕ_h^τ *are the continuous and discrete dual potentials.* $\tau, h > 0$

- Potential ϕ satisfies $m = \partial_\tau \phi$ and $\rho = 1 + L_h \phi$.

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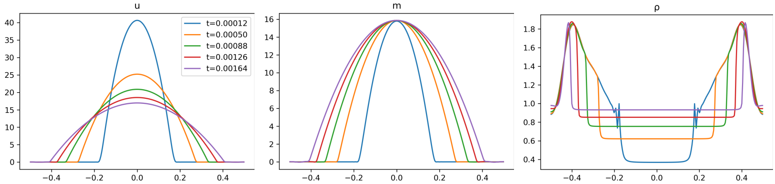
Numerical experiment, using the Barenblatt profile.

- Compactly supported, non-smooth solution of the QPME

$$u(t,x) := \frac{2}{t^\alpha} \max\left\{0, \gamma - \frac{\beta}{4}\frac{\|x\|^2}{t^{2\beta}}\right\},$$

$\alpha := \frac{d}{d+2}, \beta := \frac{1}{d+2}$. Obtained expressions of $m_T(t,x), \rho_T(t,x)$



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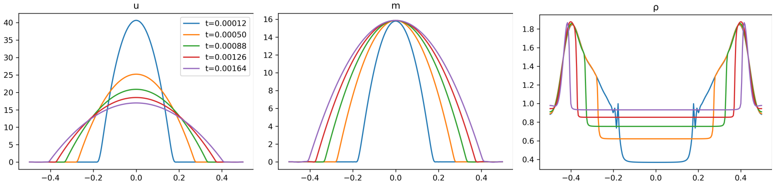
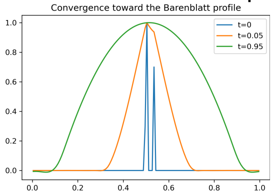


- Barenblatt profile is an attractor. BBB formulation with only 10 timesteps on $[0,1]$. (explicit $> 30\,000$ timesteps, semi-implicit $> 5\,000$ timesteps)



Multi-dimensional anisotropic QPME

- Consider a stencil $E \subseteq \mathbb{Z}^d$ and smooth weights λ^e s.t.

$$\mathcal{D}(x) = \sum_{e \in E} \lambda^e(x) e e^\top$$

- Monotone numerical scheme $-L_h u = \mathrm{div}(\mathcal{D}\nabla u) + \mathcal{O}(h^2)$,

$$-L_h u(x) = \sum_{\substack{e \in E \\ \nu = \pm}} \lambda^e(x + \tfrac{1}{2} h \nu e) \, \frac{u(x + h\nu e) - u(x)}{h^2}$$

- Same convergence result and numerical approach (primal-dual algorithm using a space-time FFT).

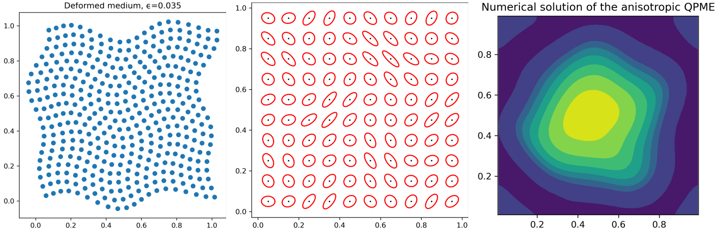


Figure: Synthetic two-dimensional experiment (deformed medium)

Voronoi/Selling decomp. of positive quadratic forms

- We use adaptive finite difference based on the decomposition of the anisotropy matrix $D \in \mathcal{S}_d^{++}$ with non-negative weights $\lambda^e \geq 0$, integer offsets $e \in \mathbb{Z}^d$.

$$\sum_{e \in \mathbb{Z}^d} \lambda^e e e^\top = D$$

F. Bonnans, G. Bonnet, M, *Monotone Discretization of Anisotropic Differential Operators Using Voronoi's First Reduction*. Constr. Approx. (2023)

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- Voronoi/Selling selection principle:

$$\max_{\lambda:\mathbb{Z}^d \to [0,\infty[} \sum_{e \in \mathbb{Z}^d} \lambda^e \quad \text{subject to} \quad \sum_{e \in \mathbb{Z}^d} \lambda^e e e^\top = D.$$

(There is a solution with $\leq d(d+1)/2$ positive coefficients.)

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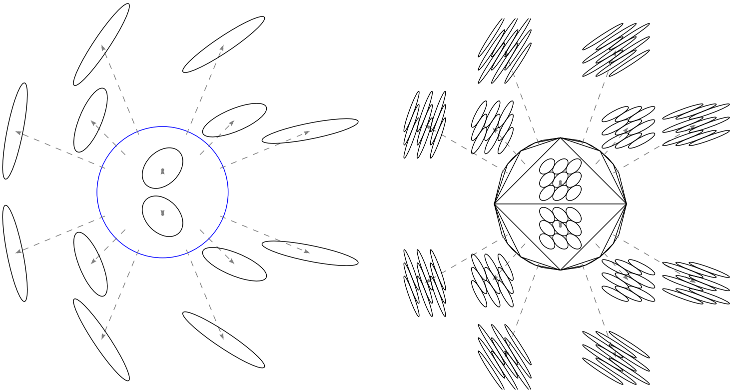
- Dual linear program:

$$\min_{M \in \mathcal{S}_d} \operatorname{Tr}(DM) \quad \text{s.t. } \forall e \in \mathbb{Z}^d \setminus \{0\},\ \|e\|_M^2 := \langle e, Me \rangle \geq 1.$$

- Periodic sphere packing pb: replace $\operatorname{Tr}(DM)$ with $\det(M)$.

F. Bonnans, G. Bonnet, M, *Monotone Discretization of Anisotropic Differential Operators Using Voronoi's First Reduction*. Constr. Approx. (2023)

- Left: Unit ball defined by $D = \begin{pmatrix} 1+a & b \\ b & 1-a \end{pmatrix}$, $a^2 + b^2 < 1$.
- Right: Linear program minimizer $2M$. Support of decomp.



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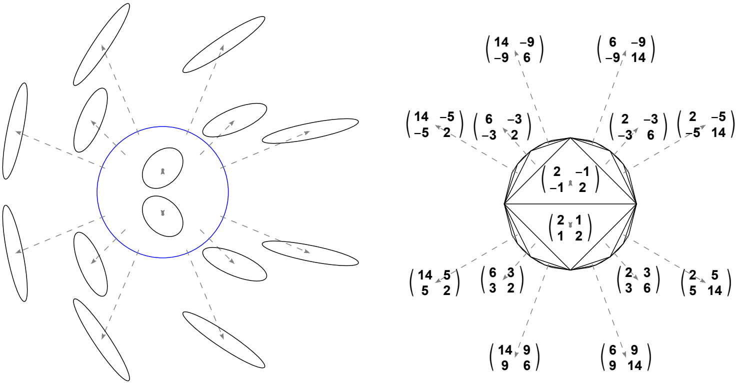

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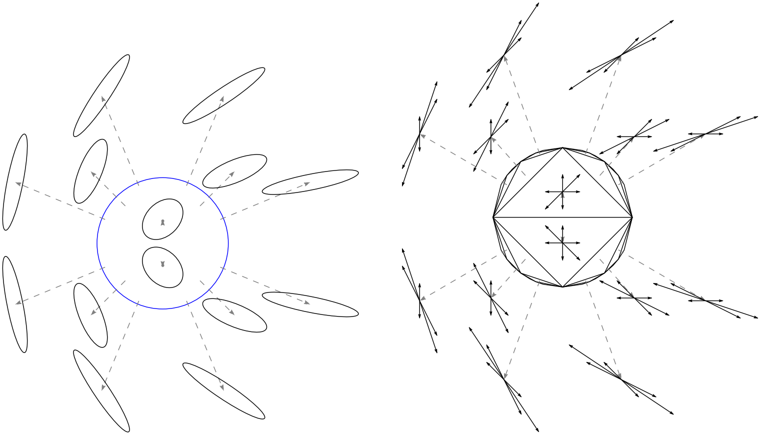

Theorem (Properties of Voronoi's decomposition)

There are computable coefficients $\lambda^e \in \mathrm{Lip}(\mathcal{S}_d^{++}, [0,\infty[)$ *s.t.*

- *(Consistency)* $D = \sum_{e \in \mathbb{Z}^d} \lambda^e(D) ee^\top$
- *(Support) cardinality:* $\#\{e \in \mathbb{Z}^d \mid \lambda^e(D) \neq 0\} \leq N(d)$, *and intrinsic radius:* $\lambda^e(D) \neq 0 \Rightarrow \|e\|_{D^{-1}} \leq R(d)\|D^{-\frac{1}{2}}\|$
- *(Spanning, if* $d \leq 4$*)* $\exists e_1, \cdots, e_d \in \mathbb{Z}^d$, $|\det(e_1, \cdots, e_d)| = 1$ *and* $\lambda^{e_1}(D), \cdots, \lambda^{e_d}(D) > 0$.
- *(Unimodular inv.)* $\lambda^e(D) = \lambda^{Ae}(ADA^\top)$, $\forall A \in \mathrm{GL}(\mathbb{Z}^d)$.

Theorem (Properties of Voronoi's decomposition)

There are computable coefficients $\lambda^e \in C^\infty(\mathcal{S}_d^{++}, [0,\infty[)$ *s.t.*

- *(Consistency)* $D = \sum_{e\in\mathbb{Z}^d} \lambda^e(D) ee^\top$
- *(Support) cardinality:* $\#\{e \in \mathbb{Z}^d \mid \lambda^e(D) \neq 0\} \leq N(d)$, *and intrinsic radius:* $\lambda^e(D) \neq 0 \Rightarrow \|e\|_{D^{-1}} \leq R(d)\|D^{-\frac{1}{2}}\|$
- *(Spanning, if* $d \leq 6$*)* $\exists e_1, \cdots, e_d \in \mathbb{Z}^d$, $|\det(e_1,\cdots,e_d)| = 1$ *and* $\lambda^{e_1}(D), \cdots, \lambda^{e_d}(D) > 0$.
- *(Unimodular inv.)* $\lambda^e(D) = \lambda^{Ae}(ADA^\top)$, $\forall A \in \mathrm{GL}(\mathbb{Z}^d)$.

Variant with smooth coefficients, obtained by considering a smooth strictly convex variant of Voronoi's linear program.

$$\max_\lambda \sum_{e\in \mathrm{supp}(\rho)} \lambda^e - \delta\rho^e\mathcal{B}\left(\frac{\lambda^e}{\rho^e}\right) \qquad \text{subject to } D = \sum_{e\in\mathrm{supp}(\rho)} \lambda^e ee^\top$$

Barrier fct $\mathcal{B}(s) := \frac{1}{2}s^2 - \ln s$. Carefully chosen weights $\rho_e(D)$.

M. Haloui, L. Métivier, M, Selling's decomposition and the anisotropic wave equation, preprint, 2025

Structure preserving *anisotropic* PDEs on *grids*

Based on Voronoi's decomp: $D = \sum_{e \in E} \lambda^e e e^\top$, $\lambda^e \geq 0$, $E \subseteq \mathbb{Z}^d$

- Causal schemes for eikonal type PDEs, $\|v\|_D := \sqrt{v^\top D v}$

$$\|\nabla u(x)\|_D^2 = \sum_{e \in E} \frac{\lambda^e}{h^2} \max\{0, u(x) - u(x-he), u(x) - u(x+he)\}^2 + \mathcal{O}(h)$$

Guillaume Bonnet, M, *Monotone discretization of the Monge-Ampère equation of optimal transport*, M2AN, 2022

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- Monotone schemes for degenerate elliptic PDEs

$$\mathrm{Tr}(D\nabla^2 u(x)) = \sum_{e\in E} \lambda^e \frac{u(x+he) - 2u(x) + u(x-he)}{h^2} + \mathcal{O}(h^2)$$

M, *Riemannian Fast-Marching on Cartesian Grids, Using Voronoi's First Reduction of Quadratic Forms*. SINUM, 2019

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- Low dispersion error scheme for wave eq. $\partial_{tt} q = \mathrm{div}(D \nabla q)$

$$\frac{q_{n+1}(x) - 2q_n(x) + q_{n-1}(x)}{\tau^2} = \sum_{\substack{e \in E \\ \nu = \pm}} \lambda^e(x + \tfrac{1}{2} h \nu e) \frac{q_n(x + h\nu e) - q_n(x)}{h^2}$$

M. Haloui, L. Métivier, M, Selling's decomposition and the anisotropic wave equation, preprint, 2025

Burgers equation : $\partial_t u + \frac{1}{2}\partial_x u^2 = \nu \partial_{xx} u, \ \ \nu \geq 0.$

BBB formulation: minimize kinetic energy among weak solutions

$$\inf_{u(0)=u_0} \sup_{\phi(T)=0} \int_{[0,T]\times\mathbb{T}} \frac{1}{2}u^2 + \underbrace{(\partial_t u + \tfrac{1}{2}\partial_x u^2 - \nu\partial_{xx} u)\phi}_{\text{Understood in the weak sense}}$$

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$$\stackrel{?}{=} \sup_{\phi(T)=0} \inf_{u(0)=u_0} \int_{[0,T]\times\mathbb{T}} \tfrac{1}{2}u^2(1-\partial_x\phi) - u(\partial_t\phi + \nu\partial_{xx}\phi) + u_0\partial_t\phi$$

Under the assumption $\partial_x\phi < 1$, and with the relations

$$(1-\partial_x\phi)u = \partial_t\phi + \nu\partial_{xx}\phi, \qquad \phi(T) = 0$$

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$$= -\inf_{\phi(T)=0} \int_{[0,T]\times\mathbb{T}} \frac{(\partial_t\phi + \nu\partial_{xx}\phi)^2}{2(1-\partial_x\phi)} - u_0\partial_t\phi$$

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$$= -\inf_{m,\rho} \int_{[0,T]\times\mathbb{T}} \frac{(m-\nu\partial_x\rho)^2}{2\rho} - u_0 m, \ \text{s.t.}\ \partial_t\rho + \partial_x m = 0, \rho(T) = 1.$$

Under the assumption $\partial_x\phi < 1$, and with the relations

$$\rho = 1 - \partial_x\phi, \qquad m = \partial_t\phi, \qquad \rho u = m - \nu\partial_x\rho.$$

- BBB formulation of Burgers' equation, $\nu \geq 0$

$$\inf_{\rho,m} \int_{[0,T]\times\mathbb{T}} \frac{(m-\nu\partial_x\rho)^2}{2\rho} - u_0 m, \text{ s.t. } \partial_t\rho + \partial_x m = 0, \rho(T) = 1.$$

- Discretized BBB energy, half timestep τ, half gridscale h

$$\sum_{\substack{t\in\mathcal{T}'_\tau\\ x\in\mathbb{T}_h}} \Big(\frac{1}{4}\sum_{\substack{\sigma_t=\pm\\ \sigma_x=\pm}} \frac{\big(m(t,x)-\nu\partial_h\rho(t+\sigma_t\tau,x)\big)^2}{2\rho(t+\sigma_t\tau, x+\sigma_x h)} - m(t,x)u_0(x)\Big),$$

subject to $\partial_\tau\rho + \partial_h m = 0$, and $\rho(T) = 1$.

- Staggered time and space grids $m \in \mathbb{R}^{\mathcal{T}'_\tau\times\mathbb{T}_h}$, $\rho \in \mathbb{R}^{\mathcal{T}_\tau\times\mathbb{T}'_h}$.

Theorem (E. Stampfli, M, 2025)

Assume a smooth positive solution on $[0,T]$, with $\nu \geq 0$. Then

$$\max_{t\in\mathcal{T}_\tau} \|\phi(t,\cdot) - \phi_{\tau h}(t,\cdot)\|_{\ell^1(\mathbb{T}_h^d)} = \mathcal{O}(\tau^2 + h^2),$$

$\phi, \phi_{\tau h}$ are the continuous and discrete dual potentials. $\tau, h > 0$

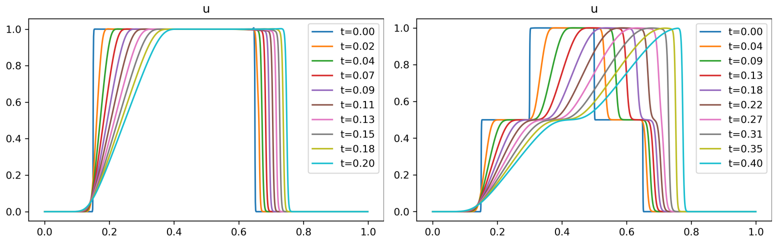
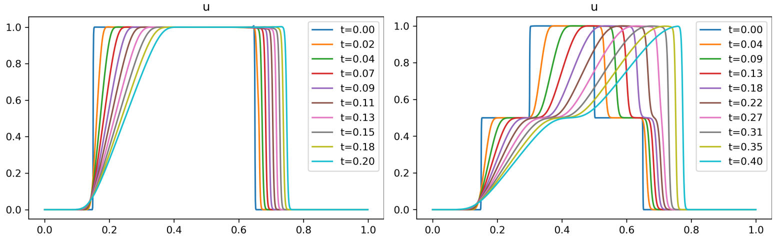


Figure: Solving Burgers equation with small viscosity, $\nu = 10^{-3}$.



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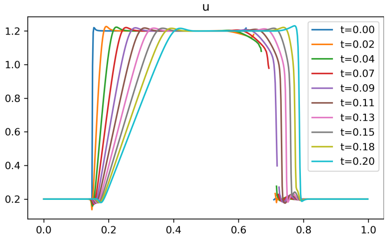

Figure: Numerical solution of inviscid Burgers, $\nu = 0$. The final time is correct, but the intermediate times are not reconstructed !

Brenier/Gallouët mountain climbing analogy

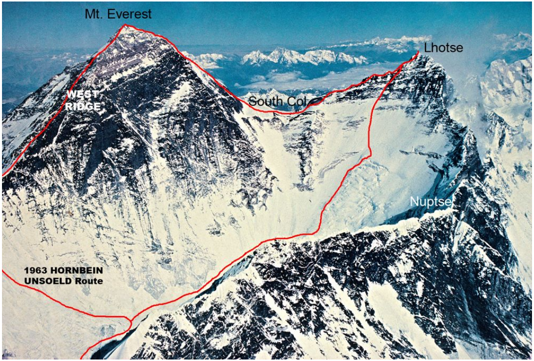

Figure: Mt. Everest to Lhotse along the crest is still an open problem.

- The BBB formulation of Burger's equation selects the weak solution with the correct final value and the fewest shocks.
- PDE ≈ path along crest. BBB ≈ path through valley.

Y. Brenier, *The initial value problem for the Euler equations of incompressible fluids viewed as a concave maximization problem*. Comm Math. Physics (2018).

Conclusion:

- BBB formulation turns evolution PDEs into global optimization problems with nice convexity properties.
- Proximal primal-dual algo, using space-time FFT. No CFL.
- Selling-based Laplacian discretization has many other uses.
- Unconventional approach: solve only the final time $u(T, \cdot)$!

Perspectives:

- Fluid mechanics PDEs.
- Weighted kinetic energy $\int_{[0,T]\times\mathbb{T}^d} e^{-\gamma t}\|u(x,t)\|^2 \mathrm{d}x\, \mathrm{d}t$.

Happy birthday and thank you Albert !

$$\mathrm{Tr}(D\nabla^2 u(x)) = \sum_{e\in E} \lambda^e \frac{u(x+he) - 2u(x) + u(x-he)}{h^2} + \mathcal{O}(h^2)$$

- Monotone discretization of Monge-Ampere, via

$$d \det(\nabla^2 u)^{\frac{1}{d}} = \inf\{\mathrm{Tr}(D\nabla^2 u) \mid D \in \mathcal{S}_d^{++}, \det D = 1\}$$

Guillaume Bonnet, M, *Monotone discretization of the Monge-Ampère equation of optimal transport*, M2AN, 2022

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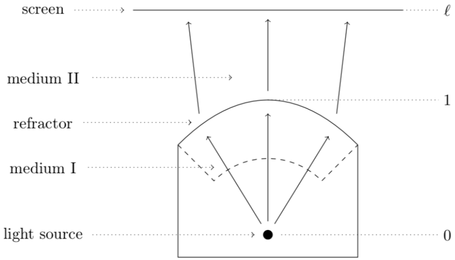

Designing a refractor projecting a given image amounts to solve

$$\det\left(\nabla^2 u(x) - A(x, \nabla u(x))\right) = B(x, \nabla u(x)),$$

$x \in \Omega$, with boundary conditions $\nabla u(x) \in P(x)$, $x \in \partial\Omega$.

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Figure: Left: image to reproduce. Right: Appelseed® render.

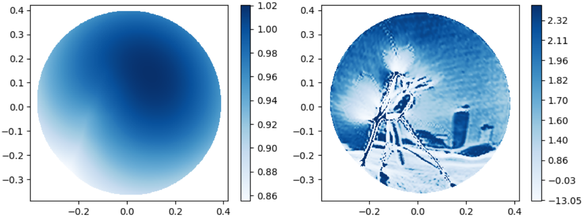

Figure: Left: refractor. Right: curvature of refractor.

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- Causal scheme (FMM solvable) for eikonal PDE $\|\nabla u\|_D = 1$
- Applications to path planning and tubular segmentation.

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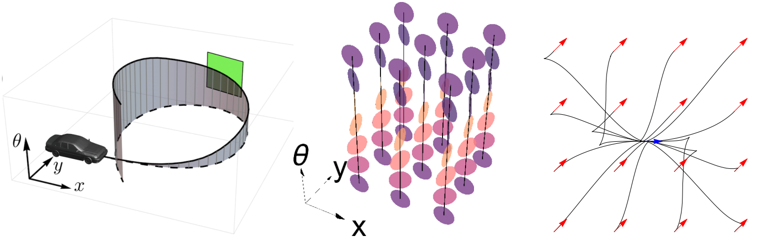
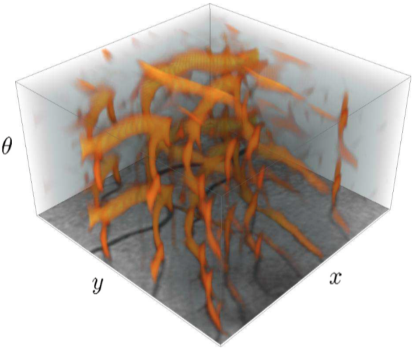
- Applications to path planning and tubular segmentation.
- Ex: Reeds-Shepp sub-Riemannian vehicle model.

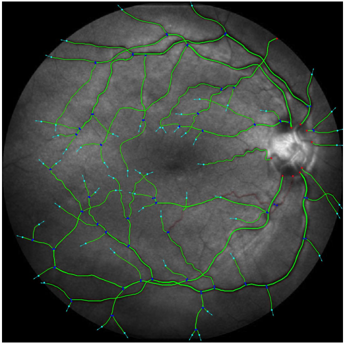


Position-orientation state space $\mathbb{M} := \mathbb{R}^2_x \times \mathbb{S}^1_\theta$, anisotropic eikonal equation with relaxation parameter $\varepsilon > 0$

$$\langle \nabla_x u, \mathrm{n}(\theta) \rangle^2 + \varepsilon^2 \langle \nabla_x u, \mathrm{n}(\theta)^\perp \rangle^2 + (\partial_\theta u)^2 = c(x, \theta)^2,$$

where $\mathrm{n}(\theta) = (\cos\theta, \sin\theta)$, and $c(x, \theta)$ is a cost function.





Left: cost function $c(x, y, \theta)$ processed from a retina scan.
Right: Reeds-Shepp vehicle minimal paths.

G. Sanguinetti, E. Bekkers, R. Duits, M. Jansen, M. Mashtakov, M, *Sub-Riemannian fast marching in* $\mathrm{SE}(2)$, Iberoamrican congress on Pattern recognition, 2015

$$\frac{q_{n+1}(x) - 2q_n(x) + q_{n-1}(x)}{\tau^2} = \sum_{\substack{e\in E\\ \nu=\pm}} \lambda^e(x+\tfrac{1}{2}h\nu e)\, \frac{q_n(x+h\nu e) - q_n(x)}{h^2}$$

- Discretizes the wave equation $\partial_{tt} q = \mathrm{div}(\mathcal{D}\nabla q)$.
- Guarantees against checkerboard artifacts (Spanning prop).
- Fourth order variant. Cv rates if smooth coefficients λ^e.

M. Haloui, L. Métivier, M, Selling's decomposition and the anisotropic wave equation, preprint, 2025

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Theorem (Reduced dispersion error of the Selling scheme)

Consider the Fourier symbol assoc. to the Selling based scheme

$$\beta_h(\xi) := \sum_{e \in E} \lambda^e \operatorname{sinc}(\tfrac{h}{2}\langle \xi, e\rangle)^2, \qquad \text{where } D = \sum_{e \in E} \lambda^e e e^\top.$$

Then for all $\xi \in \mathbb{R}^d$ and $D \in \mathcal{S}_d^{++}$,

$$|\beta_h(\xi) - \|\xi\|_D^2| \leq C(d)\, h^2\, \|\xi\|_D^4 \|D^{-1}\|.$$

Not satisfied by axis-aligned and criss-cross schemes, any $C(d)$.

M. Haloui, L. Métivier, M, Selling's decomposition and the anisotropic wave equation, preprint, 2025

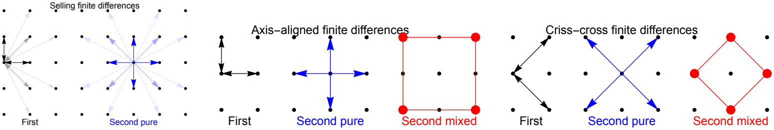


Figure: The Selling scheme avoids four-point mixed finite differences.

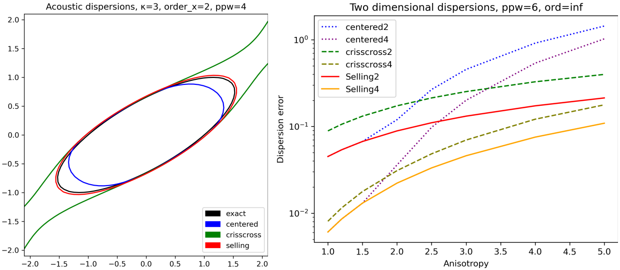
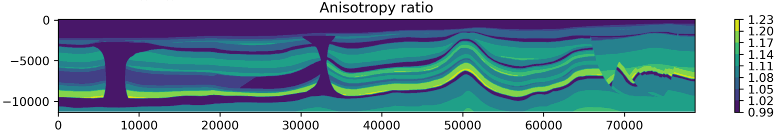
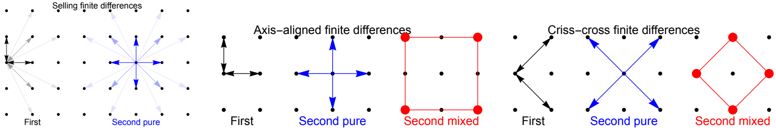


Figure: The Selling-scheme dispersion curve $\beta_h(\xi) = 1$ is closer to the ideal ellipse $\|\xi\|_D = 1$ than other schemes. Right: max error vs aniso.





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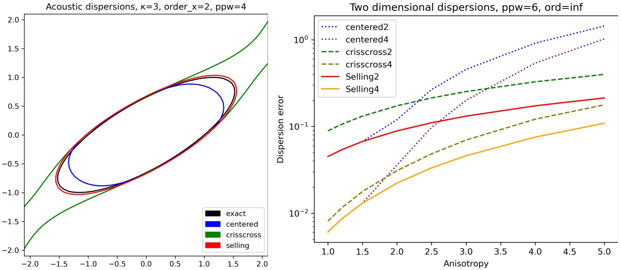

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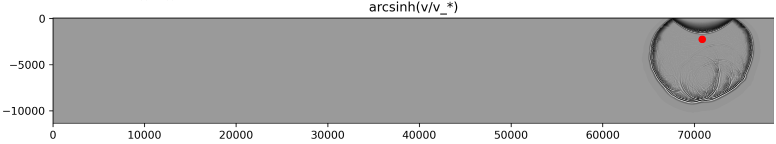
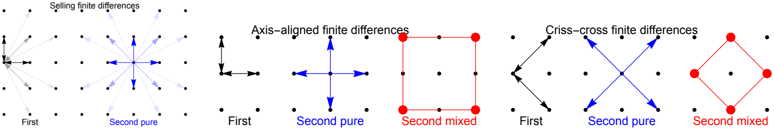


Jean-Marie Mirebeau



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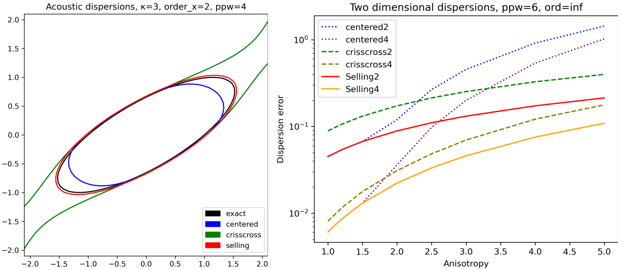

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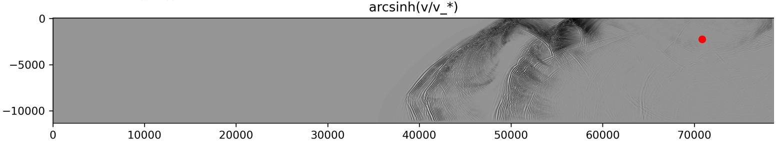

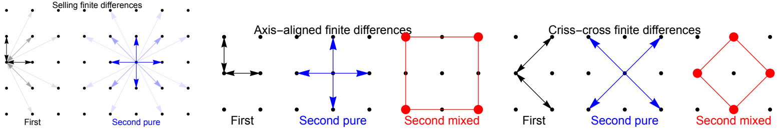

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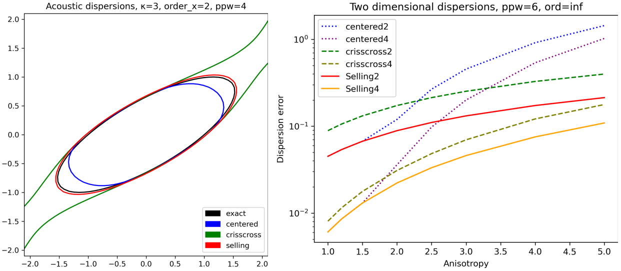

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