On the Use of Harten's MRF in optimization problems: An unfinished project ...

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Workshop in honor of Albert Cohen, Jussieu, July 2025

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4 Numerical experiments

Marc Menec, IS&3D ENG.

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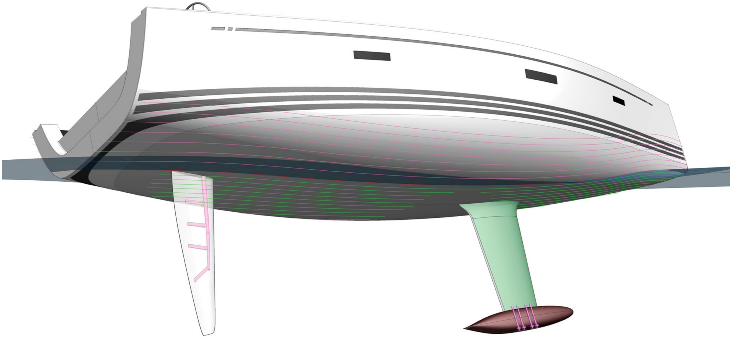
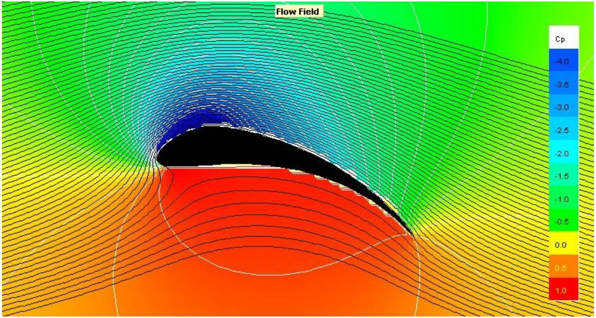


underwater appendages such as the bulb, the keel or the rudder have an important effect on performance.

Sections of appendages

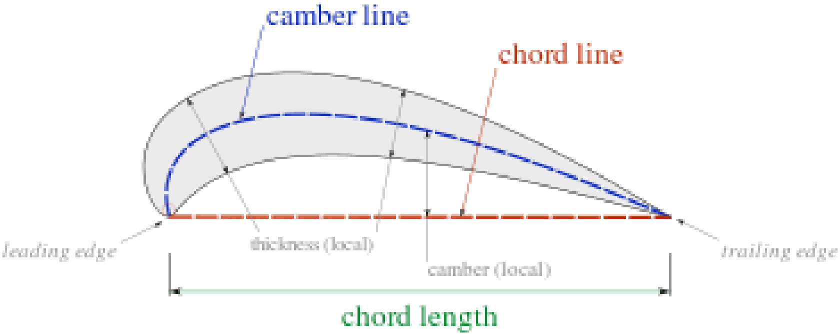
In yacht design, appendages are often constructed from a basic planar section $\alpha(t) = (x(t), y(t))$, $t \in [0, 1]$,whose shape determines the drag and lift generated by the appendage.

- the problem: search for ways to get an 'optimal' shape of a section that minimizes the drag generated by the section (while preserving some structural features).



A 2D section: $\alpha(t) = (x(t), y(t)),\ t \in [0, 1]$

$(x(0), y(0)) = (x(1), y(1))$ is the trailing edge.



In particular, the interest was to reduce the drag coefficient while preserving specific features of the section by **performing some perturbations of an original shape**

Mathematical setting in a Discrete Framework

$$\alpha \equiv (\alpha(t_i))_{i=1}^N \quad \varepsilon = (\varepsilon_i)_{i=1}^N, \quad \rightarrow \quad \alpha^\varepsilon := (\alpha(t_i) + \varepsilon_i)_{i=1}^N$$

$D(\alpha)$ (Drag Coefficient computed with *Xfoil* is a *cost function*)

Minimization Problem: $\boxed{\text{Find } \varepsilon_* \in \mathbb{R}^N : D(\alpha^{\varepsilon_*}) = \min_{\varepsilon \in \mathbb{R}^N} D(\alpha^\varepsilon)}$

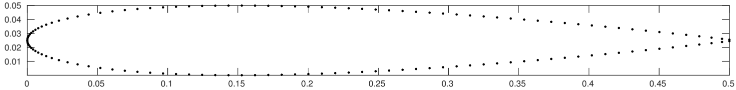
Compute a 'solution' by using an appropriate (black-box) optimizer.

Initial guess: $\varepsilon_0 = 0 \equiv \alpha^{\varepsilon_0} = \alpha$

The process is likely to be (very) slow for N moderately large .... among other problems ...

The cost may be reduced by using a multiscale strategy

Closed Curve: NACA-profile $\alpha = (x, y)$, $N = 128$ points



Required: Minimize (Reduce ...) $D(\alpha)$ (computed with *Xfoil*)

Using: Black-box minimization tools (from MATLAB):

- `fminsearch`
- `patternsearch`

Computations carried out using 'our' MSO (Multi-scale Optimization) with $L = 7$, $N_0 = 2^2$.

Reducing the drag on a size-limited foil

Initial Profile: a discrete version of the NACA0010-profile.
The aim: to locally modify it to reduce ('minimize') $D(\alpha)$ at $Re = 10^6$, while maintainig 'some' constraints.

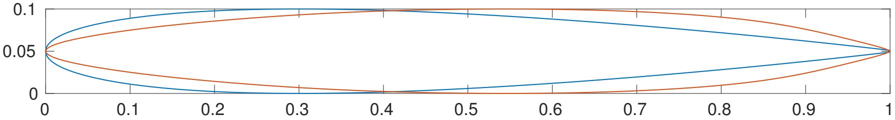
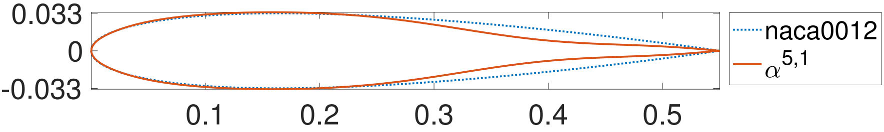


Figure: Initial profile: blue line. Output of a Multi-scale Optimization of the drag coefficient while preserving chord length and max height: red line. The drag coefficient is reduced from $9.21 \cdot 10^{-3}$ to $4.47 \cdot 10^{-3}$.

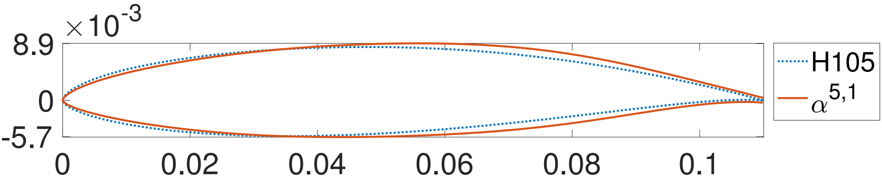
RD, S. Lopez-Ureña, M. Mennec, ECMI Proceedings 2016

Realistic simulations: Complex Cost functions and Initial shapes (blue lines) provided by IS&3D ENG.

Improving a rudder shape from an initial NACA0012 foil.



Computing an 'optimal' hydrofoil section from an initial H105 shape.



Let's try with an 'academic (Toy) example' ....

Given the grid $(t_i)_{i=0}^{2^L} = (i2^{-L})_{i=0}^{2^L}$, compute the minimum of the functional

$$F(\alpha) := \|\alpha_i - \cos(2\pi x_i)\|_2^2.$$

Minimization strategies:

- Using the MATLAB `fminsearch` function directly.
- Using the MATLAB `fminsearch` function combined with the MS-

Initial guess $y_i := \lambda \cos(2\pi x_i)$, $\qquad L = 7, N = 2^L = 128.$

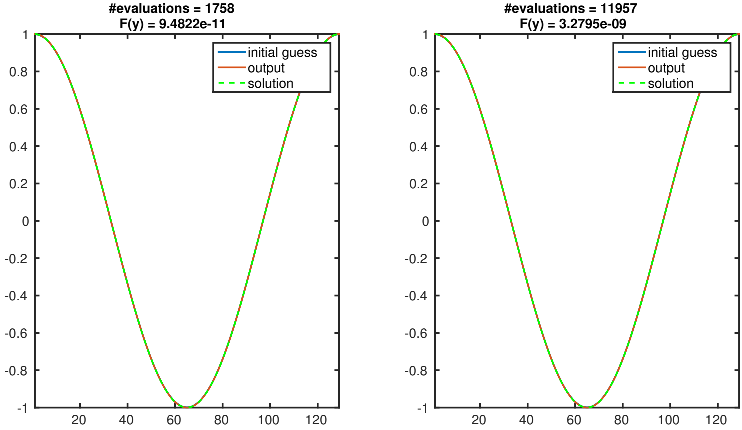
Stopping Criteria:

diff. between two consecutive iterates $< 10^{-4}$ + Cost function $< 10^{-4}$

Maximum # of iterations in `fminsearch` (function evaluations): 10^5.

Cost $\equiv$ **Number of function evaluations**.

Toy Problem

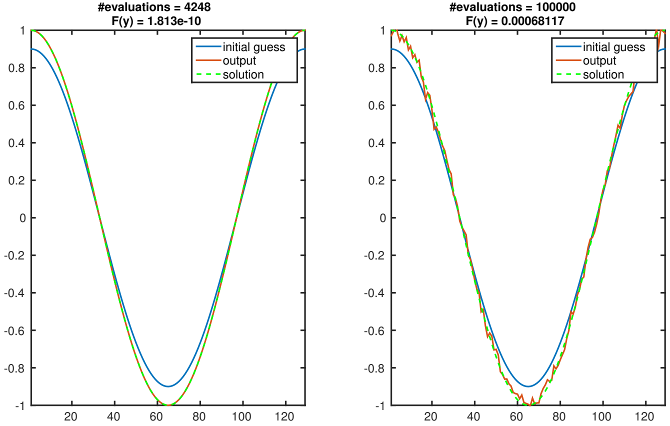


funct. eval.

$\lambda = 0.999$

MS	direct
1758	11957

Toy Problem

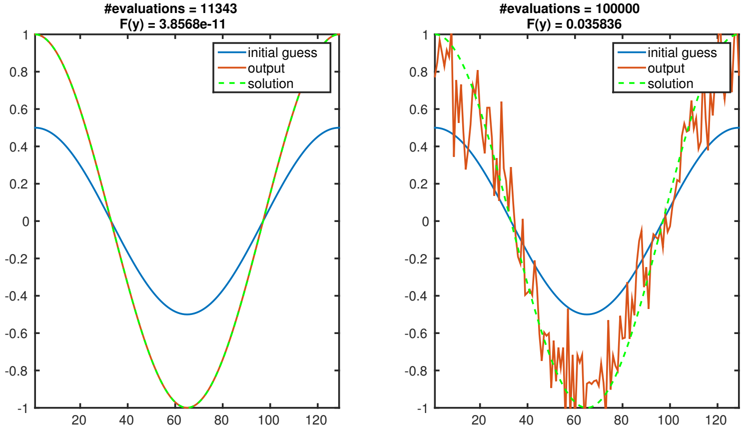


$\lambda = 0.9$

funct. eval.

MR	direct
4248	10^5

Toy Problem



$\lambda = 0.5$

\# funct. eval.

MR	direct
11343	10^5

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Discrete Multiresolution Framework

A multiresolution (MR) decomposition of a discrete data set is an equivalent representation that encodes the information as a coarse realization of the given data set plus a sequence of detail coefficients of ascending resolution.

$$\begin{array}{ccccccccc} \alpha^L & \rightarrow & \alpha^{L-1} & \rightarrow & \alpha^{L-2} & \rightarrow & \ldots \rightarrow & \alpha^0 \\ & \searrow & d^{L-1} & \searrow & d^{L-2} & \searrow & \ldots \searrow & d^0 \end{array}$$

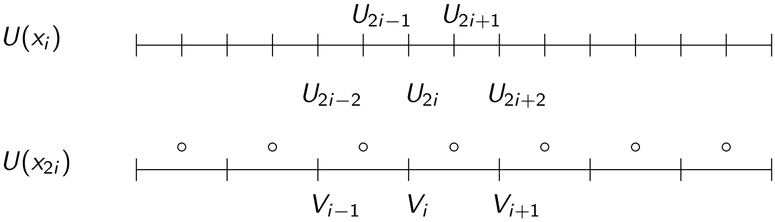
$$\alpha^L \equiv M\alpha^L = (\alpha^0, d^1, d^2, \ldots, d^L)$$

detail coefficients: difference in information between consecutive levels

Frameworks for MR:

- Wavelets (I. Daubechies, Y. Meyer, S. Mallat etc..)
- Lifting (W. Sweldens ...)
- Harten [Harten, 90's, RD, F. Arandiga A. Cohen ... 2000]
  levels of resolution: Hierarchy of nested computational meshes

Harten's Interpolatory MR framework



Decimation $\equiv$ Restriction to even values

Prediction $\equiv$ Via an interpolatory reconstruction $\mathcal{I}(x, \cdot)$

$$\left\{ \begin{array}{lcl} V_i & = & U_{2i} \\ d_i & = & U_{2i+1} - \mathcal{I}(x_{2i+1}, V) \end{array} \right\} \leftrightarrow \left\{ \begin{array}{lcl} U_{2i} & = & V_i \\ U_{2i+1} & = & \mathcal{I}(x_{2i+1}, V) + d_i \end{array} \right\}$$

$$U \in \mathbb{R}^N, \quad MU = (V, d) \overset{M\text{,2-level MRT}}{\longleftarrow - - \longrightarrow} U = M^{-1}(V, d) \quad V, d \in \mathbb{R}^{N/2}$$

Harten's Interpolatory MR framework

- **Prediction** by interpolatory reconstructions implies **Consistency** between the fine and coarse grid information:

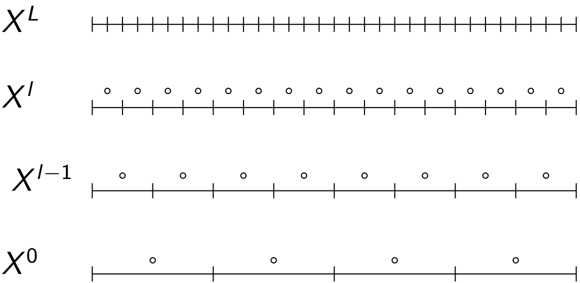
$$\tilde{U}_i = \mathcal{I}(x_i, V) \quad \Rightarrow \quad \tilde{U}_{2i} = \mathcal{I}(x_{2i}, V) = V_i = U_{2i}$$

- The **details are interpolation errors** at odd points on fine grid. Well known behavior with respect to grid-size/smoothnes of underlying data.

MR transformation: finest level X^L

$$u^L \Leftrightarrow \{u^{L-1}, d^{L-1}\} \quad \cdots \Leftrightarrow \cdots \{u^0; d^0; d^1; \cdots d^{L-1}\} = Mu^L$$

$$\begin{array}{ccccccc} u^L & \rightarrow & u^{L-1} & \rightarrow & u^{L-2} & \rightarrow & \ldots \rightarrow u^0 \\ & \searrow & d^{L-1} & \searrow & d^{L-2} & \searrow & \ldots \searrow d^0 \end{array}$$



Harten's Interpolatory MR framework

- Prediction by interpolatory reconstructions leads to (Interpolatory) Subdivision Refinement schemes
- Our notation: $\mathcal{P} = \{P_k^{k+1}\}_{k=0}^{L}$ Sequence of prediction operators between consecutive resolution levels, associated to Grids X_k with $\mathbb{N}_k$ points/relevant data. $P_k^k = I_{\mathbb{R}^{\mathbb{N}_k}}$ and

$$\text{For } 0 \leq k < l \leq L \quad P_k^l := P_{l-1}^l P_{l-2}^{l-1} \cdots P_k^{k+1} : \mathbb{R}^{\mathbb{N}_k} \longrightarrow \mathbb{R}^{\mathbb{N}_l}$$

- $\mathcal{I}(x, \cdot)$ Data-independent $\Rightarrow$ $P_k^l \in \mathbb{R}^{\mathbb{N}_l \times \mathbb{N}_k}$

  linear interpolatory subdivision schemes $\Rightarrow$ Linear MR-T.
- Our notation: MR-T between levels $k < l$, $0 \leq k < l \leq L$

$$M_{k,l} : \mathbb{R}^{\mathbb{N}_l} \longrightarrow \mathbb{R}^{\mathbb{N}_l} \qquad \text{Coarse data on } \mathbb{R}^{\mathbb{N}_k}, \text{ fine data on } \mathbb{R}^{\mathbb{N}_l},$$

- Note that if $z^k \in \mathbb{R}^{\mathbb{N}_k}$, and $z^l = P_k^l z^k \leftrightarrow M_{k,l} z^l = (z^k, 0, \ldots, 0)$

A Two-Scale ‘parameter-reduction’ approach

$$U_{\min} = \text{argmin}_{U \in \mathbb{R}^N} F(U) \quad \equiv \quad \varepsilon_* = \text{argmin}_{\varepsilon \in \mathbb{R}^N} F(U^0 + \varepsilon), \quad U^0 \in \mathbb{R}^N$$

$$U \in \mathbb{R}^N, \quad M_{0,1} U = (V, d) \overset{M\text{,2-level MRT}}{\longleftarrow - - \longrightarrow} U = M_{0,1}^{-1}(V, d) \quad V, d \in \mathbb{R}^{N/2}$$

For linear MR-T and perturbations 'at the coarse resolution level'

$$M_{0,1}^{-1}\left((V, d) + (\varepsilon^0, \vec{0})\right) = U + M_{0,1}^{-1}(\varepsilon^0, \vec{0}) = U + P_{0,1}\varepsilon^0$$

- $U^0 \in \mathbb{R}^N$ given initial gues,

  Find $\boxed{\varepsilon_*^0 = \text{argmin}_{\varepsilon^1 \in \mathbb{R}^{N/2}} F(U^0 + P_{0,1}\varepsilon^0)}$ inital guess $\varepsilon_0^1 = 0$.

  Define $U^1 := U^0 + P_0^1 \varepsilon_*^0 \quad \Rightarrow \quad F(U^1) \leq F(U^0)$

- Find $\boxed{\varepsilon_* = \text{argmin}_{\varepsilon \in \mathbb{R}^N} F(U^1 + \varepsilon) \text{ initial guess } \varepsilon_0 = 0}$

  Then $U_{\min} = U^1 + \varepsilon_*$

A Multi-scale 'parameter-reduction' approach

Find $z_{\min} \in \mathbb{R}^N$ such that $F(z_{\min}) = \min_{z\in\mathbb{R}^N} F(z)$,

Initial data (given): $\bar{z} =: z^{L,0} \in \mathbb{R}^{N_L}$,

- Find $\varepsilon_*^0 = \operatorname{argmin}_{\varepsilon^0\in\mathbb{R}^{N_0}} F(z^{L,0} + P_0^L\varepsilon^0)$, Init. guess: $\varepsilon_0^0 = \vec{0} \in \mathbb{R}^{N_0}$

  Define $z^{L,1} := z^L + P_0^L\varepsilon_*^0$, $\quad F(z^{L,1}) \leq F(z^{L,0})$

- Find $\epsilon_*^1 = \operatorname{argmin}_{\varepsilon^1\in\mathbb{R}^{N_1}} F(z^{L,1} + P_1^L\varepsilon^1)$, Init. guess: $\varepsilon_0^1 = \vec{0} \in \mathbb{R}^{N_1}$

  Define $z^{L,2} := z^{L,1} + P_1^L\varepsilon_*^1$ $\quad F(z^{L,1}) \leq F(z^{L,0})$

- ....

- Find $\varepsilon_*^L = \operatorname{argmin}_{\varepsilon^L\in\mathbb{R}^{N_L}} F(z^{L,L} + \varepsilon^L)$, Init. guess: $\varepsilon_0^L = \vec{0} \in \mathbb{R}^{N_L}$

  Define $z^{L,L+1} := z^L + \varepsilon_*^L = z_{\min}$

$$F(z^{L,L+1}) \leq F(z^{L,L}) \leq \cdots \leq F(z^{L,2}) \leq F(z^{L,1}) \leq F(z^{L,0})$$

What are we doing?

$$z^{L,0} = \bar{z} \quad M_{0,L} z^{L,0} = M_{0,L}\bar{z} = (\bar{z}^0, d^0(\bar{z}), d^1(\bar{z}), \cdots, d^{L-1}(\bar{z}))$$

$$M_{0,L}^{-1}(z_0^0 + \varepsilon^0, d^0(\bar{z}), d^1(\bar{z}), \cdots, d^{L-1}(\bar{z})) = \bar{z} + P_0^L \varepsilon^0, \ \varepsilon^0 \in \mathbb{R}^{\mathbb{N}^0}$$

$\Xi_0 := \{z^{L,0} + P_0^L \varepsilon^0, \ \varepsilon^0 \in \mathbb{R}^{\mathbb{N}^0}\}$ (affine space, $\mathbb{N}_0$ degrees of freedom)

$F_0 : \mathbb{R}^{\mathbb{N}^0} \to \mathbb{R}, \quad F_0(\varepsilon^0) = F(z^{L,0} + P_0^L \varepsilon^0), \quad \forall \varepsilon^0 \in \mathbb{R}^{\mathbb{N}^0}$

$$\varepsilon_*^0 = \text{argmin}_{\varepsilon^0 \in \mathbb{R}^{N_0}} F(z^{L,0} + P_0^L \varepsilon^0), = \text{argmin}_{\varepsilon^0 \in \mathbb{R}^{N_0}} F_0(\varepsilon^0)$$

$$z^{L,1} = z^{L,0} + P_0^L \varepsilon_*^0 \quad \to \quad z^{L,1} = \text{argmin}\{F(z), \ z \in \Xi_0\}$$

$$F(z^{L,1}) = F_0(\varepsilon_*^0) \leq F_0(0) = F(z^{L,0}) = F(\bar{z})$$

$$M_{0,L} z^{L,1} = M_{0,L} z^{L,0} + M_{0,L} P_0^L \varepsilon_*^0 = (\bar{z}^0 + \varepsilon_*^0, d^0(\bar{z}), d^1(\bar{z}), \cdots, d^{L-1}(\bar{z}))$$

What are we doing?

$$M_{0,L} z^{L,1} = (\overbrace{\bar{z}^0 + \varepsilon_*^0, d^0(\bar{z})}, d^1(\bar{z}), \cdots, d^{L-1}(\bar{z}))$$

$$z_*^1 := M_{0,1}^{-1}((\bar{z}^0 + \varepsilon_*^0, d^0(\bar{z})) = M_{0,1}^{-1}((\bar{z}^0, d^0(\bar{z})) + P_{0,1}\varepsilon_*^0 \in \mathbb{R}^{\mathbb{N}^1}$$

$$\Rightarrow M_{1,L} z^{L,1} = (z_*^1, d^1(\bar{z}), \cdots, d^{L-1}(\bar{z}))$$

and we can repeat the process with level 1 as the coarsest level.

At level k:

- $\Xi_k := \{z^{L,k} + P_k^L \varepsilon^k, \ \varepsilon^k \in \mathbb{R}^{\mathbb{N}^k}\}$ (affine space, $\mathbb{N}_k$ degrees of freedom)
- $F_k : \mathbb{R}^{\mathbb{N}^k} \to \mathbb{R}, \quad F_k(\varepsilon^k) = F(z^{L,k} + P_k^L \varepsilon^k), \quad \forall \varepsilon^k \in \mathbb{R}^{\mathbb{N}^k}$
- $\varepsilon_*^k = \text{argmin}_{\varepsilon^k \in \mathbb{R}^{N_k}} F(z^{L,k} + P_k^L \varepsilon^k), = \text{argmin}_{\varepsilon^k \in \mathbb{R}^{N_k}} F_k(\varepsilon^k)$
- $z^{L,k+1} := z^{L,k} + P_k^L \varepsilon_*^k = \arg\min\{F(z), \ z \in \Xi_k\}$

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3. Some theoretical results on MR/OPT
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Quadratic minimization problems

$$\text{Find } z_{\min} \in \mathbb{R}^N \text{ such that } F(z_{\min}) = \min_{z \in \mathbb{R}^N} F(z),$$

$$F(z) = \frac{1}{2} z^T A z - b^T z + c.$$

with A symmetric.

Proposition

If F is quadratic, and $F_k(\varepsilon^k) := F(z^{L,k} + P_k^L \varepsilon^k)$, then

$$F_k(\varepsilon^k) = \frac{1}{2} (\varepsilon^k)^T A_k \varepsilon^k - b_k^T \varepsilon_k + c_k \begin{cases} A_k & = (P_k^L)^T A P_k^L \in \mathbb{R}^{\mathbb{N}_k \times \mathbb{N}_k} \\ b_k & = (P_k^L)^T (b - A z^{L,k}) \in \mathbb{R}^{\mathbb{N}_k} \\ c_k & = F(z^{L,k}) \end{cases}$$

is quadratic. If $A \geq 0$, then $A_k \geq 0$. If $A > 0$, then $A_k > 0$.

Proposition

Let $F : \mathbb{R}^{\mathbb{N}_L} \to \mathbb{R}$ and $F_k(\varepsilon^k) = F(\hat{z} + P_k^L \varepsilon^k)$, $\hat{z} \in \mathbb{R}^{\mathbb{N}_L}$

1. If F is convex and/or $F \in \mathcal{C}^2(\mathbb{R}^{N_L}, \mathbb{R})$, then F_k is also convex and/or $F_k \in \mathcal{C}^2(\mathbb{R}^{N_k}, \mathbb{R})$.
2. If the hessian matrix $\nabla^2 F(\xi^L)$ is a positive definite matrix $\forall \xi^L \in \mathbb{R}^{N_L}$, then $\nabla^2 F_k(\xi^k)$ is a positive definite matrix $\forall \xi^k \in \mathbb{R}^{N_k}$.
3. If F is coercive, i.e. $\lim_{||z^L||_\infty \to \infty} F(z^L) = +\infty$, then F_k is coercive.

Theorem

Let $F \in \mathcal{C}^2(\mathbb{R}^{N_L}, \mathbb{R})$ be a convex coercive function such that $\nabla^2 F(\xi^L)$ is a positive definite matrix $\forall \xi^L \in \mathbb{R}^{N_L}$.

If the initial guess $z^{L,0}$ and $z_{\min} = \arg\min\{F(z),\ z \in \mathbb{R}^{N_L}\}$ can be associated to the point evaluations on $\mathcal{G}^L$ of sufficiently smooth functions, then for $0 \leq k < L$,

1. $\|z_{\min} - z^{L,k+1}\|_\infty = \mathcal{O}(h_{k+1}^{n+1})$
2. $\|z^{L,k+1} - z^{L,k}\|_\infty = \mathcal{O}(h_k^{n+1})$

where n is the degree of the interpolatory polynomials.

Summarizing ....

- Quadratic and convex cost functions: The *auxiliary problems* are of the same kind as the original problem.
- Under 'certain smoothness conditions' the distance between consecutive *sub-optimal* solutions decreases as k increases (at a rate that depends on the properties of the prediction schemes).
- Even though the auxiliary minimization problems involve an increasing number of degrees of freedom as k increases, we expect that they might be efficiently solved due to the fact that their initial guess and solution are increasingly closer.

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- We consider an 'off-the shelf' (MATLAB) optimizer $\mathcal{D}$ either `fminunc` or `patternserach`.
- Stopping criteria for the optimizer: $tol_{\mathcal{D}}$
- Stopping criteria for the MR-OPT: The max-norm of the difference between two consecutive sub-optimal solutions is less than tol_M. That is,

$$\|z^{L,k+1} - z^{L,k}\|_\infty < tol_M.$$

- $\mathcal{P}$ Interpolatory subdivision on the interval of degrees $n = 1, 3, 5$ with centered stencils on the interior and 'adjustments' at the boundaries.

Quadratic Optimization Problems: 1D

$$\begin{cases} -u''(t) + 2u(t) = f(t), & t \in (0,1) \\ u(0) = u(1) = 0. \end{cases}$$

where $f(t) := 10^6 t(1-t)(t-1/2)(t-1/4)(3/4-t)$. Using the standard centered second order discretization for u'' on a uniform grid in $[0,1]$ leads to the linear system

$$(-z_{i-1} + 2z_i - z_{i+1})J^2 + 2z_i = f(i/J), \qquad i = 1, 2, \ldots, J-1,$$

$z_0 = z_J = 0$ because of the boundary conditions, $J = N - 1$.

We compute the solution of $Ax = b$ by minimizing $F(z) = \frac{1}{2} z^T A z - b^T z$.

$$J = 128 = 2^7. \quad L = 5, \quad \mathbb{N}_L = O(10^2), \, \mathbb{N}_0 = 3 \quad tol_{\mathcal{D}} = tol_M = 10^{-6}$$

The sub-optimal solutions $z^{L,1}, z^{L,2}, z^{L,3}$

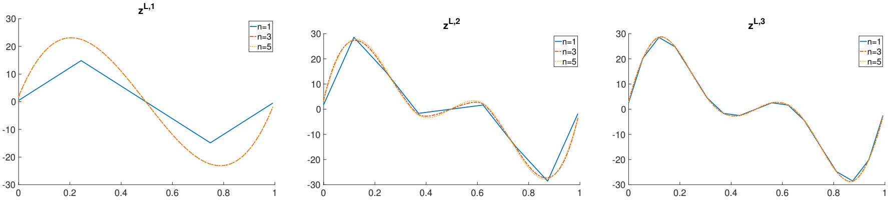

Figure: 1D BVP ($\mathcal{D} = \texttt{fminunc}$). From left to right, for $n = 1, 3, 5$.

Theoretical decay properties

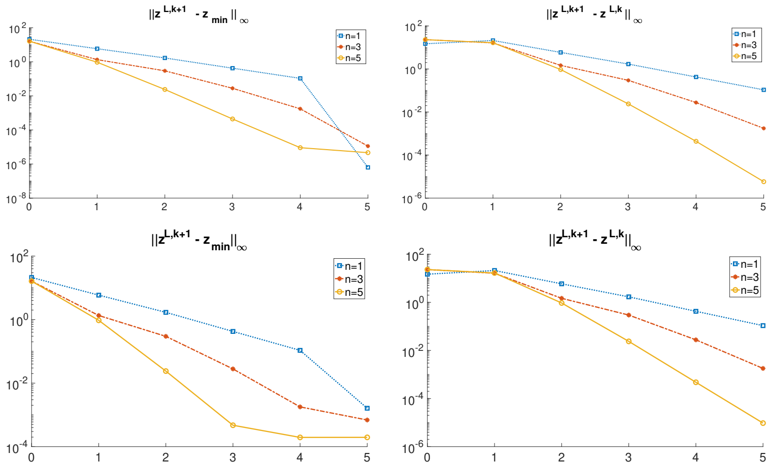

Figure: 1D BVP. Top row, $\mathcal{D} = \texttt{fminunc}$; Bottom row, $\mathcal{D} = \texttt{patternsearch}$. Horizontal axis, k (resolution level).

Theoretical decay properties

Numerical estimation of r from sub-optimal solutions.

$$r = \log_2 \frac{\|z^{L,k} - z^{L,k-1}\|_\infty}{\|z^{L,k+1} - z^{L,k}\|_\infty}.$$

$n \backslash k$	1	2	3	4	5	Theoretical rate
1	-0.51	1.84	1.80	2.00	1.98	2
3	0.48	3.54	2.47	3.37	3.80	4
5	0.45	4.24	5.33	5.78	6.09	6

Table: 1D BVP. $\mathcal{D} = \texttt{fminunc}$. (similar results with `patternsearch`)

Efficiency of MR-OPT

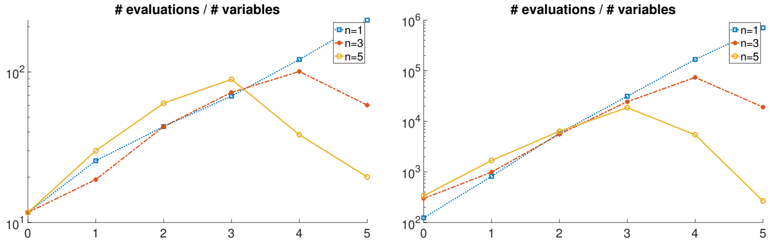

Figure: Ratio between the number of functional evaluations and the number of degrees of freedom involved in the solution of the k-th auxiliary problem, versus k. Left: $\mathcal{D} =$ `fminunc`; Right: $\mathcal{D} =$ `patternsearch`.

Table: Total number of functional evaluations required to find $z^{L,L+1}$

# F-evaluations by	$n = 1$	$n = 3$	$n = 5$	Direct-$\mathcal{D}$
`fminunc`	38 678	17 089	8 910	49 980
`patternsearch`	101 307 742	7 938 578	1 063 433	176 168 800

Quadratic Optimization Problems: 2D

$$
\begin{aligned}
-(u_{xx}(x,y) + u_{yy}(x,y)) &= f(x,y), \qquad (x,y) \in \text{int}(\Omega), \\
u(x,y) &= 0, \qquad (x,y) \in \partial\Omega,
\end{aligned}
$$

$\Omega = [0,1] \times [0,1]$, $\partial\Omega$ and $\text{int}(\Omega)$ denoting the boundary and the interior of Ω.

$$f(x,y) = \sin(4\pi x(1-x)y(1-y))$$

A standard discretization (using the classical 5-point Laplacian), on a uniform grid leads to a system of equations that can be written in matrix form as $Az = b$.

We solve the system by minimizing $F(z) = \frac{1}{2} z^T A z - b^T z$.

$$J = 128^2 \quad L = 5, \quad \mathbb{N}_L = O(10^4), \, \mathbb{N}_0 = 9 \quad tol_{\mathcal{D}} = tol_M = 10^{-7}$$

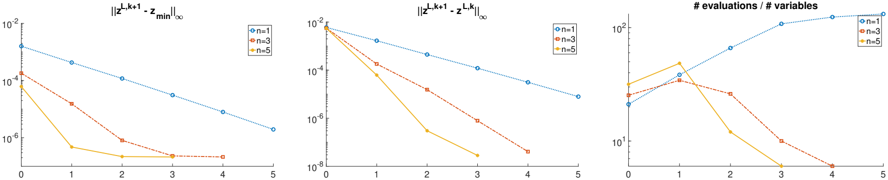

Figure: 2D Poisson problem, $L = 5$, $tol_M = tol_{\mathcal{D}} = 10^{-7}$. $\mathcal{D}$ is `fminunc`.

n	1	3	5	Direct-$\mathcal{D}$
# of F-evaluations	2 742 108	41 206	11 136	12 581 010

$n \backslash k$	1	2	3	4	5	Theoretical rate
1	1.84	1.91	1.89	1.95	1.98	2
3	5.27	3.36	4.14	4.27	-	4
5	6.72	7.59	3.62	-	-	6

Table: 2D Poisson problem. Numerical decay rate

A Non-quadratic, convex, problem: MINS

(from Frandi & Papini *Optim. Meth. & Software* 2014)

$$\min_u \int_\Omega \sqrt{1 + \|\nabla u(x,y)\|_2^2}\, d(x,y), \quad \Omega = [0,1] \times [0,1]$$

with the boundary conditions

$$u_0(x,y) = \begin{cases} x(1-x), & \text{if } y \in \{0,1\}, \\ 0, & \text{otherwise.} \end{cases}$$

Its solution is approximated by the solution of the minimization problem that results from considering as objective function

$$F(z) := \frac{1}{2N^2} \sum_{i,j=0}^{N-1} \sqrt{1 + a^2 + b^2} + \sqrt{1 + c^2 + d^2},$$

with

$$\begin{aligned} a &= N(z_{i,j+1} - z_{i,j}), & b &= N(z_{i+1,j+1} - z_{i,j+1}), \\ c &= N(z_{i+1,j+1} - z_{i+1,j}), & d &= N(z_{i+1,j} - z_{i,j}). \end{aligned}$$

$$J = 128^2 \quad L = 5, \quad \mathbb{N}_L = O(10^4), \ \mathbb{N}_0 = 9 \quad tol_{\mathcal{D}} = tol_M = 10^{-6}$$

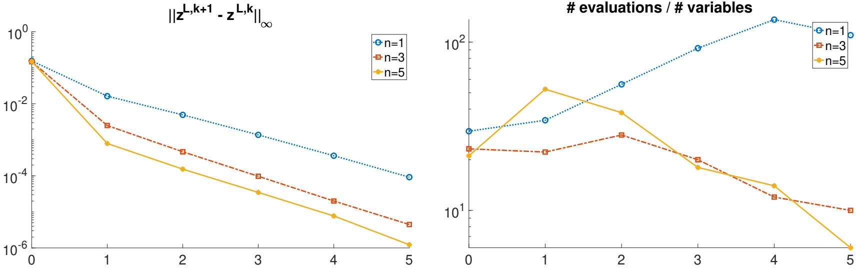

Figure: Minimal surface problem. $L = 5$, $tol_M = tol_{\mathcal{D}} = 10^{-6}$ and $\mathcal{D}$ is `fminunc`.

interpolation degree	n=1	n=3	n=5	Direct-$\mathcal{D}$
total # evaluations	2 417 132	235 771	180 990	10 226 103

Non-quadratic, non convex, problem: MOREBV

(from Frandi & Papini *Optim. Meth. & Software* 2014)

$$\begin{aligned} -(u_{xx}(x,y)+u_{yy}(x,y))+\tfrac{1}{2}(u(x,y)+x+y+1)^3 &= 0,\ (x,y)\in \mathrm{int}(\Omega), \\ u(x,y) &= 0,\ (x,y)\in \partial\Omega. \end{aligned}$$

$$\Omega = [0,1]\times[0,1]$$

Using the classical 5-point discretization of the Laplacian, the resulting system of nonlinear equations can be rewritten as a nonlinear least-squares problem with a non-convex objective function given by the expression.

$$F(z) := \sum_{i,j=1}^{N-1} \Big((4z_{i,j} - z_{i-1,j} - z_{i+1,j} - z_{i,j-1} - z_{i,j+1}) + \frac{1}{2N^2}\left(z_{i,j} + i/N + j/N + 1\right)^3 \Big)^2. \quad (1)$$

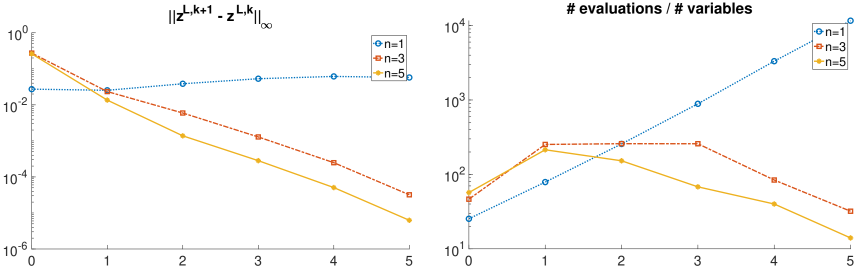

Figure: MOREBV problem. $L = 7$, $tol_M = tol_{\mathcal{D}} = 10^{-6}$ and $\mathcal{D}$ is `fminunc`.

interpolation degree	n=1	n=3	n=5	Direct-$\mathcal{D}$
total # evaluations	201 583 677	1 168 621	495 258	294 879 519

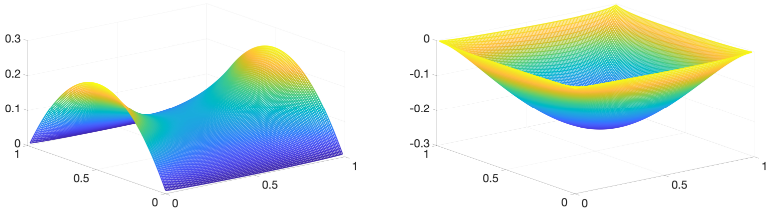

Figure: The computed solution, $z^{L,L+1}$, of the minimal surface (left) and the MOREBV (right) problems taking $n = 5$.

Conclusions : MR-OPT ...

Multilevel strategy to reduce the cost of solving optimization problems.

1. initial data provided by the user,
2. optimization tool and cost function are treated as black boxes
3. Numerical results show the efficiency of the technique

- H-MRF is used to design a sequence of *auxiliary optimization problems* that provide a finite sequence of *sub-optimal solutions*.
- Each sub-optimal solution is the initial data for the auxiliary problem at the next resolution level.
- Under some 'smoothness assumptions', we provide theoretical results that justify the efficiency of the technique. Numerical experiments show evidence.
- 'To do list ' .....

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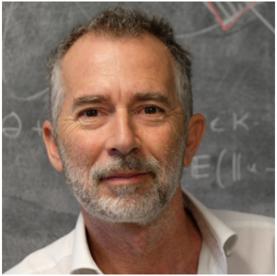

Thanks for your attention