Those were the days, my friend ...

Wolfgang Dahmen

University of South Carolina, RWTH Aachen

on the Occasion of Albert's 60th ...

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DFG-SFB 1481

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Biorthogonality and Riesz Bases

$$\Psi = \{\psi_I : I \in \mathcal{I}\}. \qquad \tilde{\Psi} = \{\psi_I : I \in \mathcal{I}\} \quad \text{(dense in } \mathbb{V}\text{)}$$

$$\langle \Psi, \tilde{\Psi} \rangle_{\mathbb{V}} = \mathbf{I}$$

Translation/dilation: $\psi_I(x) = 2^{j/2}\psi(2^j x - k),\ I \leftrightarrow (j,k),\ |I| = 2^{-j} \approx \mathrm{supp}\psi_I, \quad \psi(x) = \sum_k c_k \phi(2x-k),\ \mathbb{V} = L_2(\mathbb{R})$

A. Cohen, I. Daubechies, J.-C. Feauveau, Communications on Pure and Applied Mathematics, Vol. XLV, 485–560 (1992) – 4381

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- primal and dual multiresolution sequences: $\{\mathbb{V}_j\}, \{\tilde{\mathbb{V}}_j\}$
- room for customizations: splines, symmetry $\rightsquigarrow$ enhanced practicality
- Riesz basis property ?

$$\|\{c_I\}\|_{\ell_2} \eqsim \Big\| \sum_{I \in \mathcal{I}} c_I \psi_I \Big\|_{L_2(\mathbb{R})}$$

- exploiting Fourier-techniques

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Communities meet ...

A Game Changer

This triggered ...

- refinable functions and subdivision schemes
- biorthogonal wavelets on an interval

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- biorthogonal wavelets on bounded domains and manifolds

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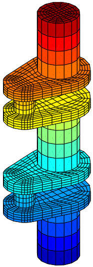

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This triggered ... the great days of the European projects ... and on Copa Cabana

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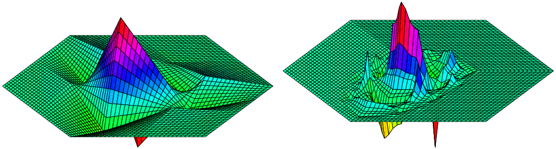

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- refinable functions and subdivision schemes
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- How about the Riesz basis property?

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Outline

Riesz Bases: a "Fourier-Free" Approach

When are biorthogonal bases Riesz bases?

$$\Psi = \{\psi_I\}_{I\in\mathcal{I}}, \quad \tilde{\Psi} = \{\tilde{\psi}_I\}_{I\in\mathcal{I}} \quad \langle \psi_I, \tilde{\psi}_{I'}\rangle = \delta_{I,I'}$$

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Multiresolution: $\mathbb{V}_j := \operatorname{span}\{\psi_I : |I|^{-\frac{1}{d}} \le j\}, \quad \tilde{\mathbb{V}}_j := \operatorname{span}\{\tilde{\psi}_I : |I|^{-\frac{1}{d}} \le j\}$

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Theorem:

$\Psi, \tilde\Psi$ are Riesz bases if $(\mathbb{V}_j)_{j\in\mathbb{N}_0}, (\tilde{\mathbb{V}}_j)_{j\in\mathbb{N}_0}$ both satisfy direct and inverse inequalities w.r.t. some modulus $\omega(\cdot, t)$

$$(J) \qquad \inf_{v_n\in\mathbb{Y}_n} \|v - v_n\|_{\mathbb{V}} \lesssim \omega(v, \rho^{-n}) \quad \mathbb{Y}_n \in \{\mathbb{V}_n, \tilde{\mathbb{V}}_n\}$$

$$(B) \qquad \omega(v_n, t) \lesssim \begin{cases} (\min\{1, t\rho^n\})^{\gamma}\|v_n\|_{\mathbb{V}}, & v_n \in \mathbb{V}_n, \\ (\min\{1, t\rho^n\})^{\tilde\gamma}\|v_n\|_{\mathbb{V}}, & v_n \in \tilde{\mathbb{V}}_n \end{cases}$$

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Important: rescaled versions of Ψ remain Riesz bases for scales of spaces “around” $\mathbb{V}$

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Besov Spaces

Wavelet characterization of function spaces:

$$\|\psi_{p,I}\|_{L_p} \eqsim 1, \quad \|\tilde{\psi}_{p,I}\|_{L_{p^*}} \eqsim 1, \quad v_I := \langle v, \tilde{\psi}_{p,I} \rangle$$

$$\|v\|_{B^s_p(L_p(\Omega))} \eqsim \left\| (|I|^{-s/d} v_I) \right\|_{\ell_p(\mathcal{I})}, \qquad 0 < s < \gamma, \;\; 0 < p < \infty$$

$\rightsquigarrow$

- Besov spaces are interpolation spaces
- Full landscape of "Sobolev" embeddings

[CDD] an unfinished book ....

BV - Correct form of Gagliardo-Nirenberg inequalities

$$|f|_{BV(\Omega)} := \sup\{\int_\Omega f \,\mathrm{div}(g) \; ; \; g \in C_c^1(\Omega, \mathbb{R}^d), \|g\|_\infty \leq 1\}$$

$\|\psi_I\|_{BV(\mathbb{R}^d)} \eqsim 1$, $f_I = \langle f, \tilde{\psi}_I \rangle$ define $\|(f_I)\|_{bv} := \|f\|_{BV(\mathbb{R}^d)}$ $\rightsquigarrow$

$$\|(f_I)\|_{bv} \lesssim \|(f_I)\|_{\ell_1} \quad \text{i.e.} \quad \ell_1(\mathcal{I}) \subset bv(\mathcal{I})$$

$\ell_2 = [\ell_\infty, \ell_1]_{1/2,2} \subset [\ell_\infty, bV]_{1/2,2} \subset [\ell_\infty, w\ell_1]_{1/2,2} = \ell_2$ $L_2 = [B^{-1}_{\infty,\infty}, BV]_{1/2,2}$

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Theorem: $d = 2$

$$\|(f_I)\|_{w\ell_1} := \sup_{\varepsilon > 0} \varepsilon \# \{ I \in \mathcal{I} : |f_I| > \varepsilon \} \le C \|f\|_{BV(\mathbb{R}^d)}$$

i.e., $\ell_1(\mathcal{I}) \subset bv(\mathcal{I}) \subset w\ell_1(\mathcal{I})$

$$\rightsquigarrow \; \|f\|_{L_2}^2 \lesssim \|f\|_{B_\infty^{-1}(L_\infty)} \|f\|_{BV}$$

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$$\|(f_I)\|_{bv} \lesssim \|(f_I)\|_{\ell_1} \quad \text{i.e.} \quad \ell_1(\mathcal{I}) \subset bv(\mathcal{I})$$

Theorem:

Let $\mathcal{R} := \{\eta \in \mathbb{R} : \eta > 1 \text{ or } \eta < 1 - 1/d\}$, then

$$\|(f_I)\|_{w\ell_1^\eta} := \sup_{\varepsilon>0} \varepsilon \Big\{ \sum_{|f_I|>\varepsilon|I|^\eta} |I|^\eta \Big\} \le C\|f\|_{BV(\mathbb{R}^d)} \qquad (1)$$

i.e., $\ell_1(\mathcal{I}) \subset bv(\mathcal{I}) \subset w\ell_1^\eta(\mathcal{I})$ iff $\eta \in \mathcal{R}$

$$\eta \in \mathcal{R},\ (s-1)p^*/d = \eta - 1,\ t = (1-\theta)s + \theta,\ \frac{1}{q} = \frac{1-\theta}{p} + \theta,\ \rightsquigarrow$$

$$\|f\|_{B_q^t(L_q(\mathbb{R}^d))} \le C\|f\|^{1-\theta}_{B_p^s(L_p(\mathbb{R}^d))} \|f\|^\theta_{BV(\mathbb{R}^d)}.$$

$q = 2,\ t = 0,\ \theta = \frac{1}{2} \rightsquigarrow p = \infty,\ s = -1 \rightsquigarrow \|f\|^2_{L_2} \lesssim \|f\|_{B_\infty^{-1}(L_\infty)} \|f\|_{BV}$

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[1] A. Cohen, W. Dahmen, I. Daubechies, R. DeVore, Tree approximation and optimal encoding, Applied and Computational Harmonic Analysis, 11 (2001), 192–226.

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[4] A. Cohen, W. Dahmen, I. Daubechies, R. DeVore, Harmonic analysis of the space BV, Revista Matematica Iberoamericana 19 (2003), 1-29.

Outline

Overarching Theme

- Problem formulation
- find performance benchmarks/measures
- constructive nonlinear solution concepts that ideally meet the benchmarks ... best *n*-term approximation, linear or nonlinear widths, Chebyshev radii ...

... Played out in a Diversity of Areas ...

- Image compression/encoding
- Mathematical learning theory
- Adaptive methods for operator equations
- Compressed sensing
- Uncertainty Quantification, high dimensional approximation
- Model reduction
- Inverse problems: state- and parameter-estimation, parameter identification

Contents

Equivalent ℓ_2-Reformulation of $Au = f$

- $A : \mathbb{V} \to \mathbb{V}'$ isomorphism
- Ψ Riesz basis for $\mathbb{V}$ (typically) a rescaled version of an L_2-Riesz basis

Theorem:

$$\boldsymbol{A} = (A\Psi)(\Psi) := \big((A\psi_I)(\psi_{I'})\big)_{I,I' \in \mathcal{I}}, \quad \mathbf{f} := f(\Psi) = (f(\psi_I))_{I \in \mathcal{I}} \Rightarrow$$

$$Au = f \;\Leftrightarrow\; \boldsymbol{A}\mathbf{u} = \mathbf{f}, \text{ and } \boldsymbol{A} : \ell_2 \to \ell_2 \text{ is an isomorphism}$$

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Proof: For $v = \sum_{I \in \mathcal{J}} v_I \psi_I =: \mathbf{v}^\top \Psi$ one has

$$\|\mathbf{v}\|_{\ell_2} \eqsim \|v\|_{\mathbb{V}} \eqsim \|Av\|_{\mathbb{V}'} \eqsim \|(Av)(\Psi)\|_{\ell_2} = \|\mathbf{v}^\top (A\Psi)(\Psi)\|_{\ell_2}$$
$$= \|\mathbf{v}^\top \boldsymbol{A}\|_{\ell_2}, \quad \mathbf{v} \in \ell_2 \quad \square$$

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- Idealized “fictitious” iteration in $\mathbb{V}$

$$(FI) \qquad \mathbf{u}^{n+1} = \mathbf{u}^n + \alpha\big(\mathbf{f} - \boldsymbol{A}\mathbf{u}^n\big), \quad n \in \mathbb{N}_0$$

- Numerical scheme: “approximate” realization of (FI)

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...at no stage is there any fixed discretization ...

Ingredients

Approximately realize: $\mathbf{u}^{n+1} = \mathbf{u}^n + \alpha(\mathbf{f} - \boldsymbol{A}\mathbf{u}^n)$

- Approximation spaces: $\mathcal{A}^s := \{\mathbf{v} \in \ell_2(\mathcal{I}) : \sigma_n(\mathbf{v}) \lesssim n^{-s}\}$

  $\sigma_{n(\varepsilon)}(\mathbf{v}) \leq \varepsilon \;\rightsquigarrow\; n(\varepsilon) \sim \varepsilon^{-1/s}$

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- Coarsening lemma: If $\|\mathbf{u} - \mathbf{v}\|_{\ell_2} \le \eta$ threshold $\rightsquigarrow \mathbf{v}_\eta$ s.t. $\|\mathbf{v} - \mathbf{v}_\eta\|_{\ell_2} \le \eta$

$$\mathbf{u} \in \mathcal{A}^s \quad \Rightarrow \quad \|\mathbf{u} - \mathbf{v}_\eta\|_{\ell_2} \le 2\eta, \quad \#\mathbf{v}_\eta \lesssim \eta^{-1/s}$$

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- Adaptive application of $\boldsymbol{A}$:

$$[AAP, \boldsymbol{A}]_j\mathbf{v} := \mathbf{w}_j := \boldsymbol{A}_j\mathbf{v}_{[0]} + \boldsymbol{A}_{j-1}(\mathbf{v}_{[1]} - \mathbf{v}_{[0]}) + \cdots + \boldsymbol{A}_0(\mathbf{v}_{[j]} - \mathbf{v}_{[j-1]}) \rightsquigarrow$$

Theorem:

$\boldsymbol{A}$ s^*-compressible, $\mathbf{v} \in \mathcal{A}^s$ $\mathbf{w}_\eta := [AAP, \boldsymbol{A}]_\eta\mathbf{v} \Rightarrow$

$$\|\mathbf{w}_\eta - \boldsymbol{A}\mathbf{v}\|_{\ell_2} \leq \eta, \quad \#\mathbf{w}_\eta, \text{ flops} \lesssim \eta^{-1/s}$$

Best n-Term Performance

Algorithm: $\mathbf{f} \mapsto \mathbf{u}(\varepsilon) \approx \boldsymbol{A}^{-1}\mathbf{f}$

- Derive accuracy-tolerances from fictitious iteration
- Compute approximate residuals using Apply $\boldsymbol{A}$
- Coarsen

Theorem:

If $\boldsymbol{A}$ is s^*-compressible, $\mathbf{u} \in \mathcal{A}^s$ ($s < s^*$), then $\mathbf{u}(\varepsilon)$ satisfies

$$\|\mathbf{u} - \mathbf{u}(\varepsilon)\|_{\mathbb{V}} \lesssim \varepsilon, \quad \text{flops}, \#\operatorname{supp} \mathbf{u}(\varepsilon) \lesssim \varepsilon^{-1/s}, \quad \|\mathbf{u}(\varepsilon)\|_{\mathbb{A}^s} \lesssim \|\mathbf{u}\|_{\mathcal{A}^s}$$

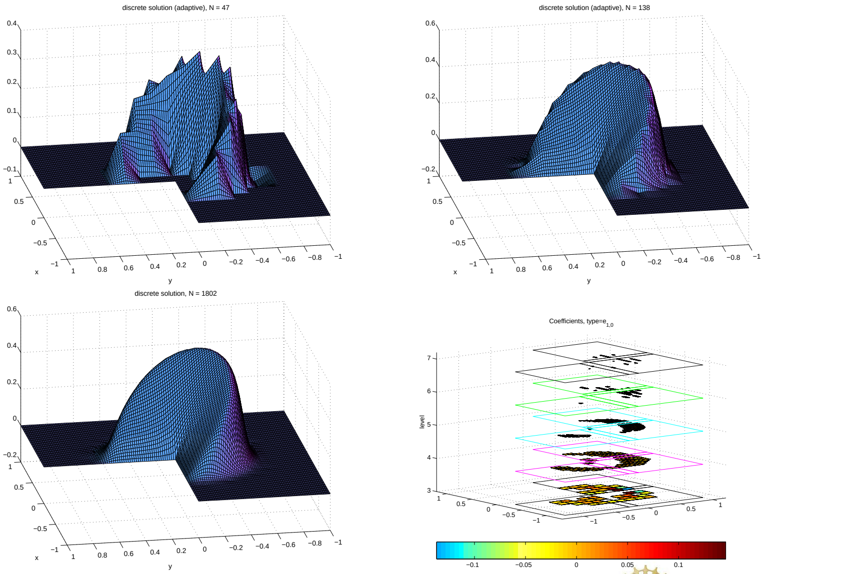

Figure: Poisson problem on an L-shaped domain

Nonlinear approximation is governed by Besov regularity

Remark:

For $d = 2$ the strongest singularity solutions (u_S, p_S) of the Stokes problem on an L-shaped domain in $\mathbb{R}^2$ belong to the scale of Besov spaces for any $s > 0$. Sobolev regularity $\leq$ 1.544483..., resp. 0.544483.... Thus arbitrarily high asymptotic rates can be obtained by adaptive schemes of correspondingly high order.... adaptivity stabilizes

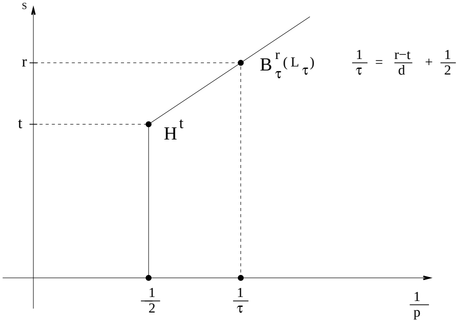

Figure: Embedding in H^t

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Stokes Problem

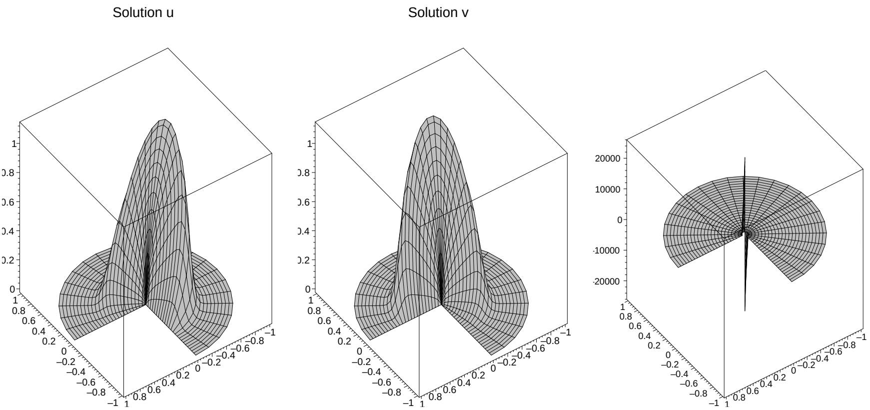

Figure: Exact solution for the first example. Velocity components (left and middle) and pressure (right). The pressure functions exhibits a strong singularity

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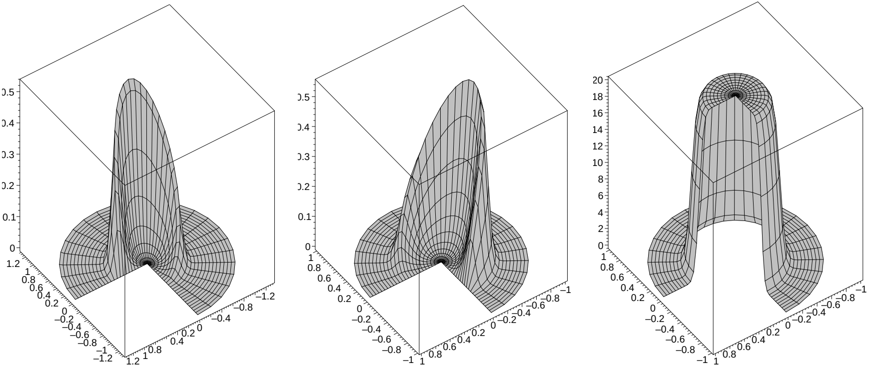

Figure: Exact solution for the second example. Velocity components (left and middle) and pressure (right).

Anything beyond?

- Indefinite and semilinear problems
- Boundary integral equations
- Adaptive eigenvalue problems
- Adaptive finite element methods
- Low-rank and tensor methods

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Contents

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A model problem in UQ

A family of uniformly elliptic problems $\mathfrak{p} \in \mathcal{Y}$

$$-\mathrm{div}(a(\mathfrak{p})\nabla u) = f \text{ in } \Omega,\ u|_{\partial\Omega} = 0, \quad 0 < r \le a(\mathfrak{p}) \le R \quad \rightsquigarrow u = u(\mathfrak{p})$$

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A single variational problem: find $u \in \mathbb{U} := L_2(\mathcal{Y}; H_0^1(\Omega))$ such that for $f \in L_2(\mathcal{Y}; H^{-1}(\Omega)) =: \mathbb{U}'$

$$a(u, v) := \int_{\mathcal{Y}} \int_{\Omega} a(\mathfrak{p})\nabla u \cdot \nabla v \, dx d\mu(\mathfrak{p}) = \int_{\mathcal{Y}} f(v) d\mu(\mathfrak{p}) =: F(v), \quad v \in \mathbb{U}$$

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- Holomorphy of $\mathfrak{p} \mapsto u(\mathfrak{p})$
- Analysis of Taylor and sparse Legendre expansions
- Summability of $(\|a_j(\cdot)\|_{L_\infty(\Omega)})_{j\in\mathbb{N}}$ in $a(x,\mathfrak{p}) = a_0(x) + \sum_{j\in\mathbb{N}} \mathfrak{p}_j a_j(x)$

  $\Rightarrow \|u - u_n\|_{\mathbb{U}} \lesssim n^{-s},\ n \in \mathbb{N}$

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Back to CDD - Low Rank and Tensor Methods for UQ

- Tensor product orthonormal (in $\mathfrak{p}$) wavelet basis for $\mathbb{U} = L_2(\mathcal{Y}; H_0^1(\Omega))$
- $a(u, v) = F(v) \quad \leftrightarrow \quad \boldsymbol{A}\mathbf{u} = \mathbf{f} \rightsquigarrow \mathbf{u}_n(x, \mathfrak{p}) = \sum_{k=1}^{n} \psi_k(x)\phi_k(\mathfrak{p})$
- Compressibility of $\boldsymbol{A}$, adaptive application of $\boldsymbol{A}$
- Coarsening lemma for tensor recompression with respect to ranks and mode representations
- Near optimal *n*-term complexity under model assumptions derived from theoretical results

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Contents

A Greedy Space Method ...Maday, Patera, Rozza, ...

$\mathbb{V}$ a Hilbert space, $\mathcal{K} \subset \mathbb{V}$ compact

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Greedy Algorithm

(i) choose $v_0 \in \mathcal{K}$, $\mathbb{V}_0 := \mathrm{span}\, v_0$

(ii) given $\mathbb{V}_n \subset \mathbb{V}$, do

$$v_{n+1} := \underset{v \in \mathcal{K}}{\mathrm{argmax}} \|v - P_{\mathbb{V}_n} v\|_{\mathbb{V}}, \quad \mathbb{V}_{n+1} := \mathrm{span}\, v_{n+1} + \mathbb{V}_n$$

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$$d_n(\mathcal{K})_{\mathbb{V}} := \inf_{\substack{\mathbb{V}_n \subset \mathbb{V} \\ \dim \mathbb{V}_n = n}} \sup_{w \in \mathcal{K}} \|w - P_{\mathbb{V}_n} w\|_{\mathbb{V}}$$

Theorem: [BCDDPW], [DPW]

$$d_n(\mathcal{K})_{\mathbb{V}} \lesssim \left\{ \begin{array}{c} n^{-\alpha} \\ e^{-cn^{\alpha}} \end{array} \right\} \quad \Rightarrow \quad \sigma_n(\mathcal{K})_{\mathbb{V}} := \sup_{u \in \mathcal{K}} \|u - P_{\mathbb{V}_n} u\|_{\mathbb{V}} \lesssim \left\{ \begin{array}{c} n^{-\alpha} \\ e^{-\tilde{c}n^{\alpha}} \end{array} \right\}$$

Practical Relevance

$\mathcal{K}$ solution manifold of a parametric PDE model: the following suffice to guarantee Kolmogorov-rate-optimality

- weak greedy concept: $v_{n+1} \in \mathcal{K}$ such that for some $\gamma > 0$

$$\inf_{v \in \mathbb{V}_n} \|v_{n+1} - v\|_{\mathbb{V}} \geq \gamma \max_{v \in \mathcal{K}} \|v - P_{\mathbb{V}_n} v\|_{\mathbb{V}}$$

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- Well posed PDEs: $\|v - u(\mathfrak{p})\|_{\mathbb{V}} \eqsim \|R(\mathfrak{p}; v)\|_{\mathbb{V}'}$, $v \in \mathbb{V}$
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- $\dim \mathcal{Y} \gg 1$: $\rightsquigarrow$ Curse of dimensionality ... remedy: for holomorphic parameter-to-solution maps trade algebraic growth of training sets against slightly weaker rates in probability
- Degenerate elliptic models - high contrast problems
- Applications to state- and parameter-estimation

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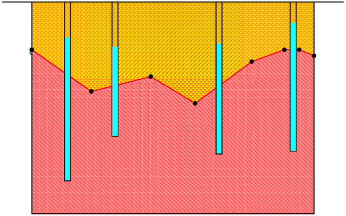
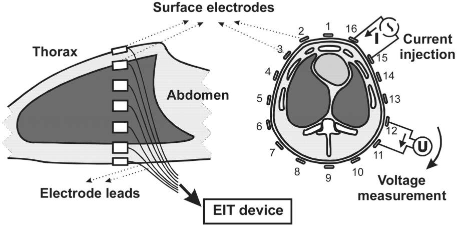
Contents

Examples





Measurements: pressure heads, voltages

Data ... and "Sensors" - PBDW

"Sensor functionals":

$$z_i = \ell_i(u), \quad \ell_i \in \mathbb{U}', \quad i = 1, \ldots, m, \ \text{fixed}$$

Y. Maday, A.T. Patera, J.D. Penn and M. Yano, *A parametrized-background data-weak approach to variational data assimilation: Formulation, analysis, and application to acoustics*, Int. J. Numer. Meth. Eng. **102**, 933-965, 2015.

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Guiding questions:

what can be achieved at best? - what are intrinsic estimation limits?

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Optimal recovery perspective, Chebyshev radii ...

Sensor coordinates $(\phi_i, v)_{\mathbb{U}} = \ell_i(v),\ v \in \mathbb{U}$

$$\mathbb{W} := \mathrm{span}\,\{\phi_i\}_{i=1}^m, \quad \mathbb{U} = \mathbb{W} \oplus \mathbb{W}^\perp, \quad \ell(u) \leftrightarrow P_{\mathbb{W}} u$$

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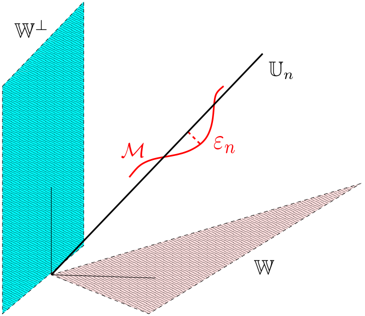
Relaxing the Prior - One-Space Method, PBDW

Suppose we have $\mathbb{U}_n \subset \mathbb{U}$, $\mathrm{dist}\,(\mathcal{M}, \mathbb{U}_n)_{\mathbb{U}} \leq \varepsilon_n$



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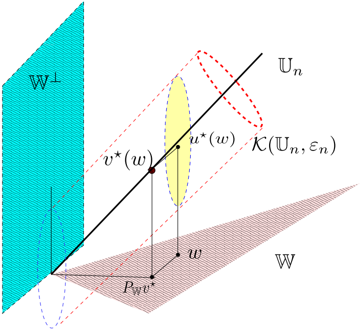
$$\mathcal{K}(\mathbb{U}_n, \varepsilon_n) := \{u \in \mathbb{U} : \mathrm{dist}\,(u, \mathbb{U}_n)_{\mathbb{U}} \leq \varepsilon_n\} \supset \mathcal{M}$$



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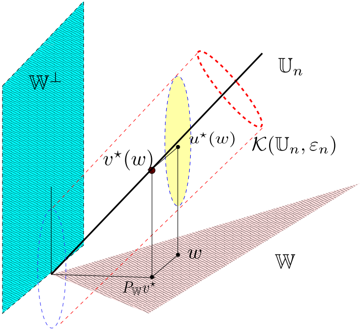

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$E_{\rm wc}(\mathcal{M}, \mathbb{W})$ is hard to achieve

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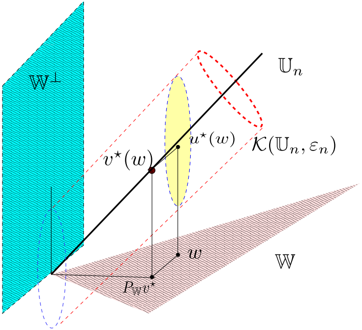

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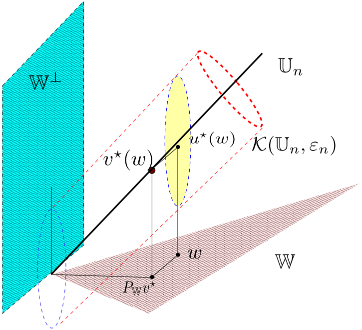

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$E_{\mathrm{wc}}(\mathcal{K}(\mathbb{U}_n, \varepsilon_n), \mathbb{W})$ is easy to achieve

$$A_{\mathbb{U}_n}(w) := u^*(w) = \underset{u \in w + \mathbb{W}^\perp}{\mathrm{argmin}} \|u - P_{\mathbb{U}_n} u\|_{\mathbb{U}}$$

Theorem:

$$\mu_n := \mu(\mathbb{U}_n, \mathbb{W}) = \max_{v \in \mathbb{U}_n} \frac{\|v\|_{\mathbb{U}}}{\|P_{\mathbb{W}} v\|_{\mathbb{U}}}$$

Then

$$\max_{u \in \mathcal{M}} \|u - u^*(P_W u)\| \le \max_{u \in \mathcal{K}} \|u - u^*(P_W u)\| = \mu_n \varepsilon_n$$

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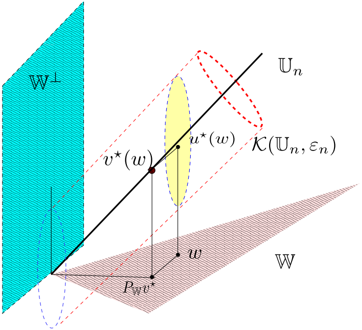

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Then

$$\max_{u \in \mathcal{M}} \|u - u^*(P_W u)\| \le \max_{u \in \mathcal{K}} \|u - u^*(P_W u)\| = \mu_n \varepsilon_n$$

Noise: $\|u - u^*(P_W u + \eta)\| \le \mu(\mathbb{U}_n, \mathbb{W})(\mathrm{dist}(u, \mathbb{U}_n) + \|\eta\|)$

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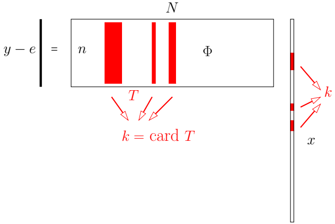
Contents

Sparse Recovery Problem

Encoder: $\Phi \in \mathbb{R}^{n\times N}$, $n \ll N$

Decoder: $\Delta : \mathbb{R}^n \to \mathbb{R}^N$

$\Sigma_k := \{z \in \mathbb{R}^N : \#\mathrm{supp}\,(z) \le k\}$



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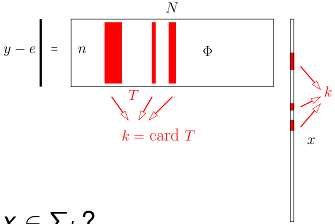

For how large k $\exists\,(\Phi, \Delta)$ s.t. $x = \Delta(\Phi x)$ for $x \in \Sigma_k$?

Instead: Best k-term approximation

$$\sigma_k(x)_{\ell_p} := \inf_{z\in\Sigma_k} \|x - z\|_{\ell_p}$$

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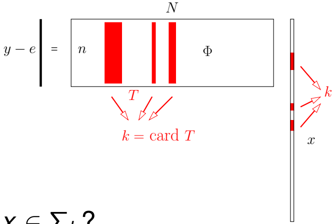

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For how large k $\exists\,(\Phi, \Delta)$ s.t. $x = \Delta(\Phi x)$ for $x \in \Sigma_k$?

Instead: Best k-term approximation

$$\sigma_k(x)_{\ell_p} := \inf_{z\in\Sigma_k} \|x - z\|_{\ell_p}$$

Question:

Given $\ell_p, N, n,$ how large can k be s.t. $\exists\,(\Phi, \Delta)$ with

$$\|x - \Delta(\Phi x)\|_{\ell_p} \le C_0\sigma_k(x)_{\ell_p}, \quad \forall\, x \in \mathbb{R}^N \quad (\mathrm{IO}(\ell_p, k)) \tag{2}$$

Known since the 70's
The Maximal Sparsity Range

$$E_{n,N}(\mathcal{K})_X := \inf_{(\Phi,\Delta)\in\mathcal{A}_{n,N}} \underbrace{\sup_{x\in\mathcal{K}} \|x - \Delta(\Phi x)\|_X}_{=:\sigma_k(\mathcal{K})_X}$$

Kashin, Gluskin/Garnaev $\rightsquigarrow$ $\sqrt{\frac{\log(N/n)+1}{n}} \sim d^n(U(\ell_1^N))_{\ell_2} \sim E_{n,N}(U(\ell_1^N))_{\ell_2} \lesssim k^{-1/2}$

$$\rightsquigarrow \quad k \leq c_0 n/\log(N/n)$$

Null Space Property and Instance Optimality

$\mathcal{N} = \mathcal{N}(\Phi)$ null space of Φ; $IO(X,k)$: $\|x - \Delta(\Phi x)\|_X \leq C_0 \sigma_k(x)_X$

Theorem:

- In essence: $\exists$ Δ such that ($IO(X,k)$) iff $\|\eta\|_X \lesssim \sigma_{2k}(\eta)_X, \quad \eta \in \mathcal{N}$
- for $X = \ell_1^N$ RIP $\Longrightarrow$ Null Space Property

Restricted isometry property - RIP(k,δ)

$$(1-\delta)\|z\|_{\ell_2} \leq \|\Phi z\|_{\ell_2} \leq (1+\delta)\|z\|_{\ell_2}, \quad z \in \Sigma_k$$

A Subtle Dependence on Norms

Theorem: $X = \ell_1^N$:

Let Φ satisfy RIP$(3k, \delta)$, $\delta < \delta_0$ and $\Delta(y) := \mathrm{argmin}_{\Phi z = y} \|z\|_{\ell_1}$

$$\Rightarrow \quad \|x - \Delta(\Phi x)\|_{\ell_1} \leq C(\delta)\sigma_k(x)_{\ell_1} \quad \text{i.e., } (\Phi, \Delta) \text{ is } \mathrm{IO}(\ell_1, k)$$

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Theorem: $X = \ell_2^N$

(Φ, Δ) is IO$(\ell_2, 1)$ $\Longrightarrow$ $n \geq aN$.

A Subtle Dependence on Norms

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Let Φ satisfy RIP$(3k, \delta)$, $\delta < \delta_0$ and $\Delta(y) := \operatorname{argmin}_{\Phi z = y} \|z\|_{\ell_1}$

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Theorem: $X = \ell_2^N$

(Φ, Δ) is $\mathrm{IO}(\ell_2, 1) \implies n \geq aN$.

BUT: IOP "in probability" is feasible in ℓ_2

Theorem:

Let Φ from a family of random matrices that satisfy RIP of order $2k$ and BP with high probability. Then $\exists\, \Delta$ such that for each $x \in \mathbb{R}^N$, drawing Φ, yields

$$\|x - \Delta(\Phi x)\|_{\ell_2} \leq C_0\sigma_k(x)_{\ell_2}, \quad k \lesssim n/\log(N/n) \quad \text{with high probability}$$

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National Science Foundation

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RWTH AACHEN UNIVERSITY

Stability Limits Performance ...

$\mathbb{X}$ Banach space

$$\delta_n(\mathcal{K})_{\mathbb{X}} := \inf_{E,D} \sup_{v\in\mathcal{K}} \|v - D(E(v))\|_{\mathbb{X}}, \quad D, E \text{ subject to constraints}$$

- D, E continuous $\rightsquigarrow$ manifold widths
- D, E Lipschitz $\rightsquigarrow$ stable widths
- Carl's Inequality: $e_n(\mathcal{K})_{\mathbb{X}} = \inf\{\varepsilon > 0 : \mathcal{N}_\varepsilon(\mathcal{K}) \le 2^n\}$

  $D, D\circ E$ Lipschitz, $E(\mathcal{K})$ bounded $\rightsquigarrow$

$$\sup_{n\in\mathbb{N}} n^r(\log_2 n)^{-r} e_n(\mathcal{K})_{\mathbb{X}} \le C \sup_{n\in\mathbb{N}} n^r \delta_{\gamma,n}(\mathcal{K})_{\mathbb{X}}.$$

- Operator Learning ...

- A. Cohen, R. DeVore, G. Petrova, P. Wojtaszczyk, Optimal stable nonlinear approximation, Found. Comput. Math., 22 (2022), pp. 607–648.
- R. DeVore, R. Howard, C. Micchelli, Optimal non-linear approximation, Manuscripta Math. 4 (1989), 469–478.
- G. Petrova, P. Wojtaszczyk, Limitations on approximation by deep and shallow neural networks,Journal of Machine Learning Research 24 (353), 1–38 Dec 2022,

Operator Learning ...

- no gain for classical smoothness classes
- perhaps more to come on model classes defined by structural sparsity ... better describing solution manifolds
- modifications?

$$\delta_n(\mathcal{S},\mathcal{K})_{\mathbb{X}_{\mathcal{K}},\mathbb{U}} := \inf_{\substack{E:\mathcal{K}\to\mathbb{R}^n \\ D:\mathbb{R}^n\to\mathbb{U}}} \max_{v\in\mathcal{K}} \|\mathcal{S}(v) - D(E(v))\|_{\mathbb{U}},$$

- Manifold widths of solution manifolds of high-dimensional transport equations avoid the Curse of Dimensionality

W. Dahmen, Compositional Sparsity, Approximation Classes, and Parametric Transport Equations, Constructive Apoproximation, 61 (2025), 219–283.

Those are the days, my friend ...

... and many more to come, santé ! ...

HAPPY BIRTHDAY !