Elliptic PDEs: The Boundary Conditions Escape Plan

Andrea Bonito
Department of Mathematics • Texas A&M University

Joint with

Peter Binev	University of South Carolina
Albert Cohen	Sorbonne Université
Wolfgang Dahmen	RWTH Aachen
Alan Demlow, Ronald DeVore, Joshua Siktar and Guergana Petrova	Texas A&M University
Diane Guignard	University of Ottawa

"PDEs are made by God, the boundary conditions by the Devil !" Alan Turing

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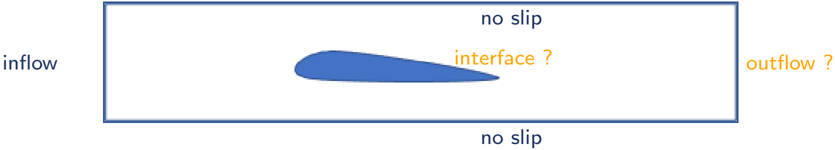
Why missing boundary information?

- Correct physics unknown
- Values not accessible
- Modified by numerical schemes

Airfoil simulation



OUTLINE

Optimal Recovery

Elliptic PDE

Stokes

Recovery Problem - Mathematical Setting

Core Problem:

We are given data observation of an unknown function $\widetilde{f} : \Omega \to \mathbb{R}$ with $\Omega \subset \mathbb{R}^d$.
We want to use this information to create $\hat{f}$ that predicts $\widetilde{f}$ away from the data.

Measure of success:

$\widetilde{f} \in X$ and we measure the success in some function norm $\|.\|_X$
e.g. $X = H^1(\Omega)$.
$\|\hat{f} - \widetilde{f}\|_X$ is the recovery error.

Data:

Measurements $w_i := l_i(\widetilde{f})$, $i = 1, ..., m$, where $l_i \in X^\#$ with $\|l_i\|_{X^\#} = 1$, e.g.

$$l : f \mapsto c \int_\Omega f(p) e^{-\frac{1}{2}|p|^2} dp \in H^1(\Omega)^\#.$$

Model Class Assumption:

Additional information about f is needed;
$\widetilde{f} \in K \subset X$, K is a compact subset of X, e.g., $K = U(H^t(\Omega))$, $t > 1$;
K is called the model class or prior.

Benchmark - Optimal Recovery [a]

Information: $K_w := \{f \in K \ : \ l_j(f) = w_j, \ j = 1, ..., m\}$.

Encodes all the information about $\widetilde{f}$.

Chebyshev Ball: Let $B = B(z_w, R_w)$ be a smallest ball in X that contains K_w.

Optimal Recovery: $\hat{f} = z_w$ is a Chebyshev center.

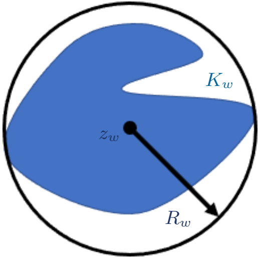
Optimal Recovery Error: $R_w := R(K_w)_X$ is the Chebyshev radius.

Near Optimal Recovery Error: We would be satisfied with $\hat{f}$ such that

$$\|\hat{f} - f\|_X \leq C R(K_w)_X, \qquad f \in K_w,$$

with a reasonable and known $C \geq 1$.

$C = 1$ when $\hat{f}$ is a Chebychev center.
$C \leq 2$ when $\hat{f}$ is any element in K_w.



[a] [Micchelli and Rivlin (1977)],[Novak and Wozniakowski (2008)],[Traub and Wozniakowski (1980)],[Binev, B., DeVore, Petrova (2024)]

OUTLINE

Model Problem

Domain: Ω bounded with Lipschitz boundary Γ.

PDE:

$$\widetilde{u} \in X := H^1(\Omega) \ : \qquad -\Delta \widetilde{u} = 0 \qquad \text{in } \Omega, \qquad \widetilde{u}|_\Gamma = \, ? \qquad \text{on } \Gamma.$$

Model Class - Boundary Regularity: $\widetilde{u}|_\Gamma \in H^{1/2}(\Gamma) \Longleftrightarrow \widetilde{u} \in H^1(\Omega)$. We assume that $\widetilde{u}$ satisfies for some $s > \frac{1}{2}$

$$\widetilde{u}|_\Gamma \in U(H^s(\Gamma)), \qquad \text{i.e.,} \qquad \|\widetilde{u}\|_{H^s(\Gamma)} \leq 1.$$

$$K := \{v \in H^1(\Omega) \ : \ \Delta v = 0, \ \|v\|_{H^s(\Gamma)} \leq 1\} \subset\subset H^1(\Omega).$$

Unit Ball Property:

$$\mathcal{H}^s := \{v \in H^1(\Omega) \ : \ \Delta v = 0, \quad v|_\Gamma \in H^s(\Gamma)\} \subset H^1(\Omega).$$

is a Hilbert space with $\|v\|_{\mathcal{H}^s} = \|v\|_{H^s(\Gamma)}$.

$$K = U(\mathcal{H}^s)$$

Measurements:

$$l_i(\widetilde{u}) = w_i \in \mathbb{R}, \ i = 1, ..., m \quad \text{given, with } l_i \in U(H^1(\Omega)^\#).$$

Taking advantage of the Hilbertian Setting

Riesz Representers of Measurements: $l_i \in H^1(\Omega)^\# \subset (\mathcal{H}^s)^\#$

$$\varphi_i \in \mathcal{H}^s \quad : \qquad \langle \varphi_i, v\rangle_{H^s(\Gamma)} = l_i(v), \qquad \forall v \in \mathcal{H}^s.$$

Information: $W := \mathrm{span}(\varphi_1, ..., \varphi_m)$, the information $l_i(v) = w_i \iff$ the information $P_W v = v_W$.

$$K_w = \{U(\mathcal{H}^s) \ : \ l_i(v) = w_i, \quad i = 1, ..., m\} = U(\mathcal{H}^s) \cap \{P_W \widetilde{u} + w^\perp, \ w^\perp \in W^\perp\}$$

Chebychev Radius: [Micchelli and Rivlin (1977)],

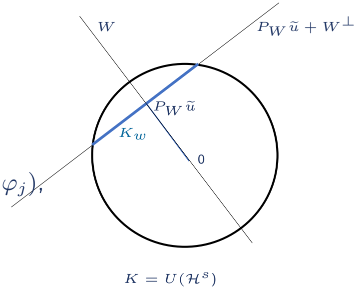
$$z_w = \arg\min_{v \in K_w} \|v\|_{\mathcal{H}^s}$$

$$z_w = P_W \widetilde{u} = \sum_{j=1}^m z_j \varphi_j,$$

$\mathbf{w} := (w_1, ..., w_m)^T$, $G_{ij} := \langle \varphi_i, \varphi_j\rangle_{H^s(\Gamma)} = l_i(\varphi_j)$,

$\mathbf{z} := (z_1, ..., z_m)^T$

$G\mathbf{z} = \mathbf{w}$ (Moore-Penrose)



Unit Ball Property: $K = U(\mathcal{H}^s)$ where $\mathcal{H}^s$ is a Hilbert space with $\|v\|_{\mathcal{H}^s} = \|v\|_{H^s(\Gamma)}$.

Numerical Algorithm

Given $\epsilon > 0$

Step 1: For $j = 1, ..., m$, compute an approximation $\widehat{\varphi}_j \in H^1(\Omega)$ of φ_j such that

$$\|\varphi_j - \widehat{\varphi}_j\|_{H^1(\Omega)} \leq \varepsilon.$$

Recall that $\varphi_j \in \mathcal{H}^s$ is defined by

$$\langle \varphi_j, v \rangle_{H^s(\Gamma)} = l_j(v), \qquad \forall v \in \mathcal{H}^s.$$

Step 2: Define $\widehat{G} = (l_i(\widehat{\varphi}_j))_{i,j=1}^m$ and find a coefficient $\boldsymbol{\alpha}$ such that

$$\widehat{G}\boldsymbol{\alpha} = \mathbf{w}.$$

We compute the Moore-Penrose inverse with thresholding.

Step 3: Assemble the recovery function

$$\widehat{u} = \sum_{j=1}^m \alpha_j \widehat{\varphi}_j.$$

Theorem [Binev, B., Cohen, Dahmen, DeVore, Petrova (2024)]

For any $\varepsilon > 0$, the approximate recovery function

$$\widehat{u} = \sum_{j=1}^{m} \alpha_j \widehat{\varphi}_j$$

satisfies for all $u \in K_w$ (and in particular $\widetilde{u}$)

$$\|u - \widehat{u}\|_{H^1(\Omega)} \leq R(K_w)_{H^1(\Omega)} + C\varepsilon,$$

for a constant C mainly depending on the condition number of G^{-1} and m (explicit formula). Equivalently

$$\|u - \widehat{u}\|_{H^1(\Omega)} \leq C(\varepsilon) R(K_w)_{H^1(\Omega)},$$

with $C(\varepsilon) \to 1^+$ as $\varepsilon \to 0^+$.

Near Optimal Recovery

Key Ingredient: PDE for the Riesz Representers $\varphi := \varphi_i$

Continuous Problem:

$$\varphi \in \mathcal{H}^s \quad : \qquad \langle \varphi, v \rangle_{H^s(\Gamma)} = l(v), \qquad \forall v \in \mathcal{H}^s.$$

Trace Formulation: $\varphi \in \mathcal{H}^s \implies \varphi = E\psi$ is the harmonic extension of $\psi \in H^s(\Gamma)$.

Fractional Diffusion Problem: Find $\psi \in H^s(\Gamma)$ such that

$$\langle \psi, \eta \rangle_{H^s(\Gamma)} = l(E\eta) =: \ell(\eta), \qquad \forall \eta \in H^s(\Gamma).$$

Equivalent Formulation: We choose $\|(I - \Delta_\Gamma)^{s/2} . \|_{L_2(\Gamma)}$ as norm on $H^s(\Gamma)$, $s < s^*$.

Find $\psi \in H^s(\Gamma)$, the weak solution of

$$(I - \Delta_\Gamma)^s \psi = \ell \in H^{-s}(\Gamma), \qquad \text{where} \qquad (I - \Delta_\Gamma)^s \psi(x) := \sum_{i=1}^{\infty} \lambda_i^s (\psi, b_i)_{L_2(\Gamma)} b_i(x).$$

Resolvant Formula $(0 < s < 1)$**:**

$$\psi = \frac{1}{2\pi i} \int_{\mathcal{C}} z^{-s} (zI - (I - \Delta_\Gamma))^{-1} \ell \, dz,$$

where $\mathcal{C} \subset \mathbb{C} \setminus (-\infty, 0] \times \{0\}$ is an oriented curve with the eigenvalues of $(I - \Delta_\Gamma)$ to its right.

Key Ingredient: PDE for the Riesz Representers $\varphi := \varphi_i$

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Balakrishnan Representation $(0 < s < 1)$**:**

$$\psi = \frac{\sin(\pi s)}{\pi} \int_0^\infty t^{-s} (tI + (I - \Delta_\Gamma))^{-1} \ell \, dt = \frac{\sin(\pi s)}{\pi} \int_{-\infty}^\infty e^{(1-s)y} (e^y I + (I - \Delta_\Gamma))^{-1} \ell \, dy.$$

For $s > 1$, iterate $(I - \Delta_\Gamma)^s = (I - \Delta_\Gamma)^{s-1} (I - \Delta_\Gamma)$ for $1 < s < 2$, ...

Finite Element on Surfaces for Fractional Diffusion Problems

Subdivision: $\mathcal{T}_h$ of Ω assumed to be Lipschitz and polygonal

Finite Element Spaces: $\mathbb{V}_h := \mathbb{V}(\mathcal{T}_h)$ space of continuous pw linear functions, $\mathbb{T}_h = \mathbb{V}_h|_\Gamma$ space of continuous pw linear functions (on Γ).

Discrete Harmonic Extension: $E_h : \mathbb{T}_h \to \mathbb{V}_h$ is defined for $w_h \in \mathbb{T}_h$ by

$$\int_\Omega \nabla E_h w_h \cdot \nabla v_h = 0, \qquad \forall v_h \in \mathbb{V}_h \cap H^1_0(\Omega); \qquad E_h w_h|_\Gamma = w_h.$$

RHS approximation: $\ell_h(v_h) := l(E_h v_h)$.

Galerkin: $\psi_h \in \mathbb{T}_h$ is given by

$$\psi_h = \frac{\sin(\pi s)}{\pi} \int_{-\infty}^{\infty} e^{(1-s)y} w_h(y)\, dy,$$

where $w_h(y) \in \mathbb{T}_h$ approximates $(e^y I + (I - \Delta_\Gamma))^{-1}\ell$ and is given by

$$(1+e^y)\int_\Gamma w_h(y) v_h + \int_\Gamma \nabla_\Gamma w_h(y) \cdot \nabla_\Gamma v_h = \ell_h(v_h), \qquad \forall v_h \in \mathbb{T}_h.$$

Sinc Quadrature: Given N, set $k \approx \lceil 1/\sqrt{N} \rceil$ and $y_j = jk$, $j = -N, ..., N$,

$$\psi_h^N := \frac{\sin(\pi s)}{\pi} k \sum_{j=-N}^{N} e^{(1-s)y_j} w_h(y_j), \qquad \varphi_h^N := E_h \psi_h^N.$$

PDE Regularity Property[a]: Let $t^* > 0$ be the largest number so that for $0 \le t \le t^*$ we have

$$(I - \Delta_\Gamma)^{-1} : H^{-1+t}(\Gamma) \to H^{1+t}(\Gamma)$$

is an isomorphism and

$$E : H^{\frac{1}{2}+t}(\Gamma) \to H^{1+t}(\Omega)$$

is bounded.

Approximate Riesz Representers: We have that $\varphi = E\psi$ and $\varphi_h^N := E_h \psi_h^N$ satisfy

$$\|\varphi - \varphi_h^N\|_{H^1(\Omega)} \lesssim (h^\beta + e^{-\pi^2\sqrt{N}})\|l\|_{H^1(\Omega)^\#},$$

where $\beta := \min(2s - 1, t^* + \frac{1}{2}, 2t^*)^- > 0$ is "optimal" for FEM on quasi-uniform subdivisions.

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[a] [Grisvard (2011)] and [Buffa, Costabel, Schwab (2002)]

[b] [B., Guignard and Lee (2024)] based on [B. and Lee (2022)] and [B. and Pasciak (2015)]

Error Estimates - Optimal Recovery

Set $\hat{u}_h^N := \sum_{j=1}^m \hat{U}_j \varphi_{j,h}^N$ where $\mathbf{U} = (\hat{U}_1, ..., \hat{U}_m)$ satisfies $\widehat{G}\mathbf{U} = \mathbf{w}$, and $\widehat{G}_{ij} = l_i(\varphi_{j,h}^N)$. There exists C (depending on Ω, s, and r) such that

$$\sup_{u \in K_w} \|u - \hat{u}_h^N\|_{H^1(\Omega)} \leq R(K_w)_{H^1(\Omega)} + C(h^\beta + e^{-\pi^2\sqrt{N}}),$$

$\hat{u}_h^N$ approximate the Chebysev center

Near optimal Recovery [B. and Guignard (2024)]

$$\sup_{u \in K_w} \|u - \hat{u}_h^N\|_{H^1(\Omega)} \leq C(h, N) R(K_w)_{H^1(\Omega)}$$

provided h is sufficiently small and N is sufficiently large (depending on the measurements) and where $C(h, N) \to 1^+$ as $h \to 0^+$ and $N \to \infty$.
Apply approximation estimate from [Binev, B., Cohen, Dahmen, DeVore, Petrova (2024)].

In particular for the targeted solution to the PDE $\widetilde{u}$

$$\|\widetilde{u} - \hat{u}_h^N\|_{H^1(\Omega)} \leq C(h, N) R(K_w)_{H^1(\Omega)}$$

What about pointwise measurements?[a]

Defined on the Model Class: $l(v) = v(\overline{x})$, $\overline{x} \in \overline{\Omega}$, is a linear functional on $\mathcal{H}^s$ provided $\mathcal{H}^s \subset C^0(\overline{\Omega})$, i.e.,

$$s > (d-1)/2 \qquad \Longrightarrow \qquad \text{additional restriction when } d = 3$$

The algorithm requires:

$$\|\varphi - \widehat{\varphi}_h\|_{H^1(\Omega)} + \|\varphi - \widehat{\varphi}_h\|_{L^\infty(\Omega)} \leq \varepsilon.$$

Finite Element Error when $d = 2$ and $s = 1$:

$$\|\varphi - \widehat{\varphi}_h\|_{L^\infty(\Omega)} \lesssim h^{\min(\frac{1}{2}, 2t^*)}$$

and $0 < t^* \leq 1$ (elliptic regularity on Γ and continuity of extension operator)

[a] [Binev, B., Cohen, Dahmen, DeVore, Petrova (2024)]

Pointwise measurement located inside Ω[a]

Advantage: We avoid the requirement $\mathcal{H}^s \subset C^0(\overline{\Omega})$ since the functions in $\mathcal{H}^s$ are harmonic inside Ω.

No additional requirement on s when $d = 3$.

Finite Element Error when $d = 2$ and $s = 1$:

$$\|\varphi - \widehat{\varphi}_h\|_{L^\infty(\Omega)} \lesssim h^{\min(\frac{3}{2}, t^*) + \min(\frac{5}{2}, t^*)}$$

where $0 < t^* \leq 1$ (elliptic regularity on Γ and continuity of extension operator).

Improved rate

Tools: Local pointwise estimates based on the distance to the boundary.

Drawback: The constant in the error estimate blows up as $d(\overline{x}, \Gamma) \to 0$.

[a] [B., Demlow, Siktar (soon)]

OUTLINE

Optimal Recovery

Elliptic PDE

Stokes

Setting

System of Equations: The velocity $\mathbf{u} : \Omega \to \mathbb{R}^d$ and the pressure $p : \Omega \to \mathbb{R}$ are related via the system of equations

$$-\Delta \mathbf{u} + \nabla p = 0 \qquad \text{and} \qquad \operatorname{div} \mathbf{u} = 0 \qquad \text{in } \Omega.$$

Ambient space:

$$X := H^1(\Omega)^d \times L_2(\Omega), \qquad \|\mathbf{v}, q\|_X^2 := \|\mathbf{v}\|_{H^1(\Omega)^d}^2 + \|p\|_{L_2(\Omega)}^2.$$

Model class:

$$K := \left\{ (\mathbf{v}, p) \in H^1(\Omega)^d \times L_2(\Omega) \ : \ \|\mathbf{u}\|_{H^s(\Gamma)^d} \leq 1, \qquad \int_\Omega p = 0 \right\}.$$

Measurements: $l_i(\mathbf{u}, p)$ in general, for the numerical illustration we consider measurements on $\mathbf{u}_i$ or p separately.

Optimal Recovery for Stokes[b]

Chebyshev Radius: Depends on the approximation of the couple velocity-pressure in an intricate way.

$$\sup_{(\mathbf{u},p)\in K_\omega} \|\mathbf{u}-\widehat{\mathbf{u}}, p-\widehat{p}\|_X = R(K_\omega)_X, \qquad (\widehat{\mathbf{u}},\widehat{p}) \text{ Chebyshev center.}$$

Optimal recovery algorithm: Same as for the Poisson problem but with the harmonic extension replaced by the "Stokes" extension.

Quantity of interest: We consider the drag on $\gamma \subset \Gamma$ with normal $\boldsymbol{\nu}$

$$Q(\mathbf{v},q) := \int_\gamma \mathbf{e}_1^T \left(\nabla \mathbf{v} + \nabla \mathbf{v}^T - qI\right) \boldsymbol{\nu}.$$

Following [a], its value $Q(\widehat{\mathbf{u}},\widehat{p})$ at Chebyschev center is its optimal recovery

$$\sup_{(\mathbf{u},p)\in K_\omega} |Q(\widehat{\mathbf{u}},\widehat{p}) - Q(\mathbf{u},p)| = \inf_{\alpha\in\mathbb{R}} \sup_{(\mathbf{u},p)\in K_\omega} |\alpha - Q(\mathbf{u},p)|$$

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[a] [Foucart and Hengartner (2025)]

[b] [B. and Guignard (in progress)]

Comments on the mean value condition - in progress

Model class: Consists of pressures with $\int_\Omega p = 0$

General pressure: The targeted pressure may not have vanishing mean value
$\Longrightarrow$ the measurements may not correspond to a pressure with vanishing mean value;

Optimal Recovery: Depend on the pressure recovery (average) in an even more intricate way.

Questions:

- Is it possible to construct a practical algorithm to recover the mean value of the targeted pressure from the measurements? Ok when $l_i(\mathbf{u}, p) = l_i(p) = p(x_i)$[a].

- Is it possible to incorporate $\int_\Omega p = C$ (unknown C) directly in the recovery algorithm?

[a] [Foucart, Hielsberg, Mullendore, and Petrova (2019)]

Setup I

Domain: Square domain $\Omega = (0,1)^2$;

Functions to Recover:

$$\mathbf{u} = \begin{pmatrix} e^x \cos(y) \\ -e^x \sin(y) + 2x^2 \end{pmatrix}, \quad p(x,y) = 2(2y-1), \qquad s = 1$$

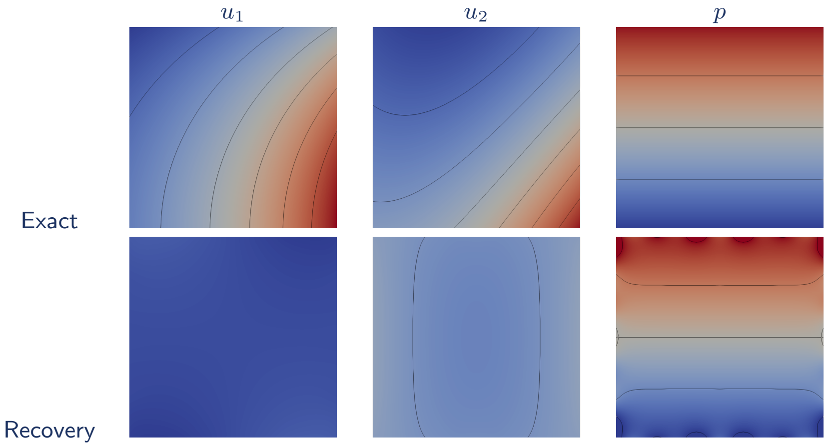
Measurements:

$$l_j(v) = \frac{1}{\sqrt{2\pi r^2}} \int_\Omega \exp\left(-\frac{|x - z_j|^2}{2r^2}\right) v(x) \mathrm{d}x$$

with $r = 0.1$ and uniformly distributed $z_j \in \Omega$, $j = 1, 2, \ldots, m$.

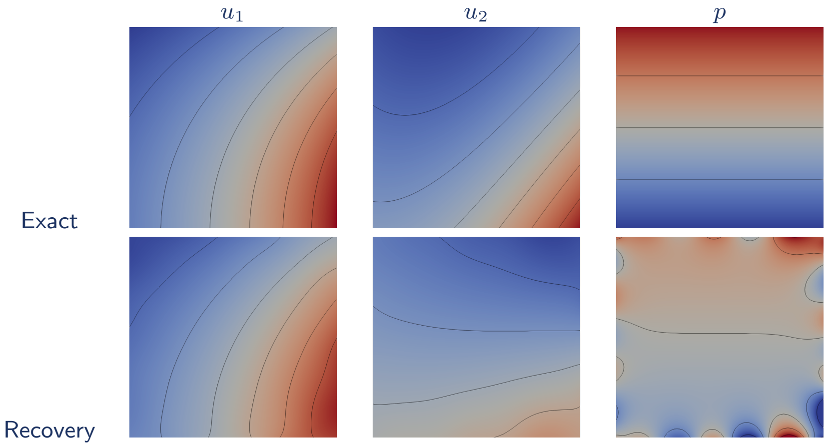
Discretization Parameters: h sufficiently small.

Preliminary results - Square domain



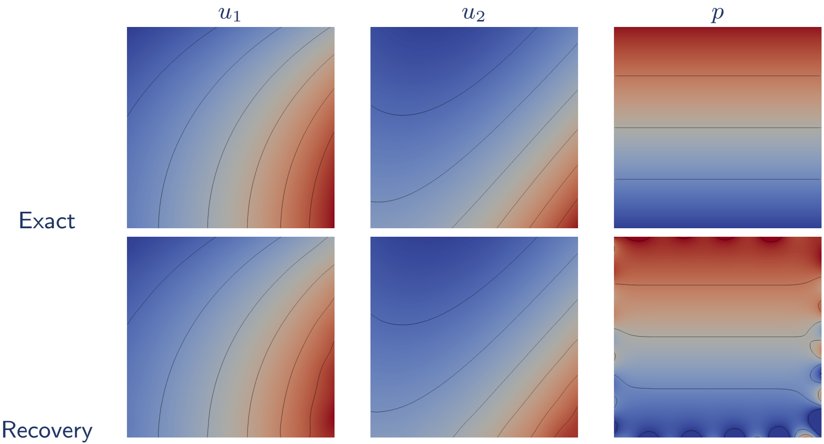
$$(m_{u_1}, m_{u_2}, p) = (0, 0, 36), \qquad \|\mathbf{u} - \hat{\mathbf{u}}\|_{H^1(\Omega)^2} = 3.1281, \qquad \|p - \hat{p}\|_{L_2(\Omega)} = 0.0848.$$

Preliminary results - Square domain



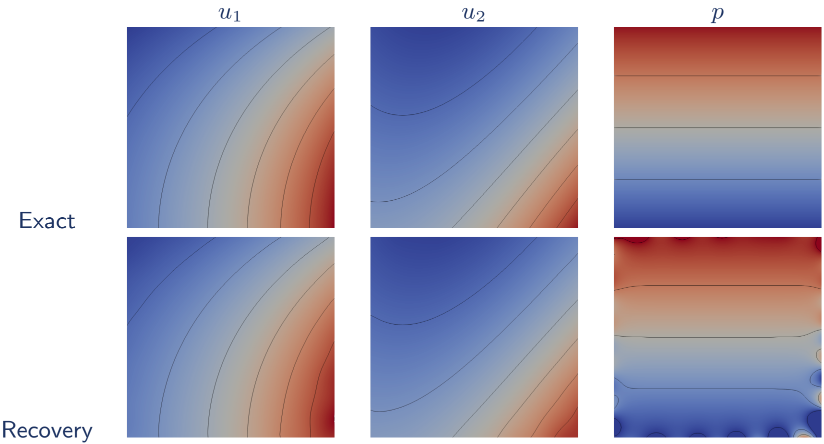
$$(m_{u_1}, m_{u_2}, p) = (36, 0, 0), \qquad \|\mathbf{u} - \hat{\mathbf{u}}\|_{H^1(\Omega)^2} = 1.5936, \qquad \|p - \hat{p}\|_{L_2(\Omega)} = 0.8731.$$

Preliminary results - Square domain



$$(m_{u_1}, m_{u_2}, p) = (36, 36, 0), \qquad \|\mathbf{u} - \hat{\mathbf{u}}\|_{H^1(\Omega)^2} = 0.2865, \qquad \|p - \hat{p}\|_{L_2(\Omega)} = 0.2381.$$

Preliminary results - Square domain



$$(m_{u_1}, m_{u_2}, p) = (36, 36, 36), \qquad \|\mathbf{u} - \hat{\mathbf{u}}\|_{H^1(\Omega)^2} = 0.2851, \qquad \|p - \hat{p}\|_{L_2(\Omega)} = 0.2337.$$

Preliminary results - Square domain

Velocity recovery error $\|\mathbf{u} - \hat{\mathbf{u}}\|_{H^1(\Omega)^2}$:

m_u \ m_p	0	1	4	9	16	25	36	49	64
0	–	3.274	3.166	3.138	3.132	3.128	3.128	3.129	3.129
1	3.181	3.181	2.814	2.780	2.777	2.773	2.774	2.774	2.775
4	1.773	1.766	1.621	1.025	0.821	0.729	0.729	0.735	0.737
9	1.050	1.038	0.943	0.894	0.708	0.513	0.420	0.386	0.385
16	0.632	0.629	0.617	0.552	0.496	0.476	0.325	0.271	0.272
25	0.412	0.410	0.398	0.406	0.402	0.338	0.308	0.248	0.228
36	0.287	0.286	0.287	0.283	0.278	0.299	0.285	0.228	0.218
49	0.238	0.237	0.237	0.237	0.244	0.261	0.233	0.227	0.214
64	0.203	0.202	0.197	0.189	0.206	0.181	0.220	0.216	0.209

$$\|\mathbf{u}\|_{H^1(\Omega)^2} = 3.2739$$

Preliminary results - Square domain

Pressure recovery error $\|p - \hat{p}\|_{L_2(\Omega)}$:

m_u \ m_p	0	1	4	9	16	25	36	49	64
0	–	1.155	0.623	0.287	0.277	0.128	0.085	0.050	0.046
1	2.485	2.485	0.662	0.308	0.279	0.129	0.085	0.050	0.046
4	1.425	1.422	1.229	0.799	0.497	0.242	0.143	0.083	0.072
9	0.925	0.901	0.703	0.658	0.494	0.346	0.207	0.118	0.100
16	0.507	0.499	0.485	0.388	0.398	0.365	0.243	0.164	0.134
25	0.336	0.332	0.319	0.327	0.312	0.277	0.250	0.178	0.152
36	0.238	0.237	0.238	0.235	0.233	0.247	0.234	0.173	0.167
49	0.196	0.195	0.194	0.196	0.199	0.215	0.194	0.187	0.169
64	0.165	0.163	0.165	0.148	0.156	0.151	0.186	0.182	0.169

$$\|p\|_{L_2(\Omega)} = 1.1547$$

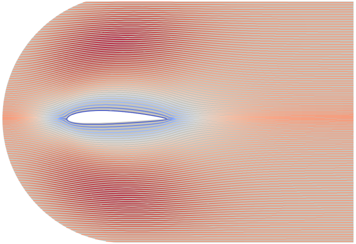
Preliminary results - Square domain

Total recovery error $\left(\|\mathbf{u}-\hat{\mathbf{u}}\|^2_{H^1(\Omega)^2}+\|p-\hat{p}\|^2_{L_2(\Omega)}\right)^{1/2}$:

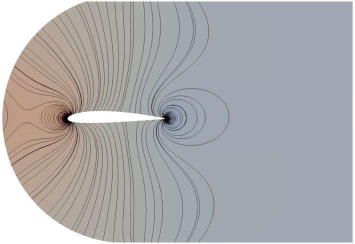
m_u \ m_p	0	1	4	9	16	25	36	49	64
0	–	3.472	3.227	3.151	3.145	3.131	3.129	3.129	3.129
1	4.037	4.037	2.890	2.797	2.791	2.776	2.775	2.775	2.775
4	2.275	2.267	2.035	1.299	0.960	0.768	0.743	0.740	0.740
9	1.400	1.375	1.176	1.110	0.863	0.619	0.468	0.404	0.398
16	0.811	0.803	0.784	0.675	0.636	0.600	0.406	0.317	0.303
25	0.531	0.528	0.510	0.521	0.509	0.437	0.397	0.306	0.274
36	0.373	0.372	0.372	0.367	0.363	0.388	0.369	0.286	0.274
49	0.308	0.307	0.306	0.307	0.315	0.338	0.303	0.294	0.273
64	0.262	0.260	0.257	0.240	0.258	0.236	0.288	0.283	0.268

$$\left(\|\mathbf{u}\|^2_{H^1(\Omega)^2}+\|p\|^2_{L_2(\Omega)}\right)^{1/2}=3.4716$$

velocity (streamlines)



pressure (isocontours)



Reference solution boundary conditions:
$\mathbf{u} = \mathbf{0}$ on the airfoil, $(\nabla \mathbf{u}^T + \nabla \mathbf{u} - pI)\mathbf{n} = \mathbf{0}$ on the outflow, $\mathbf{u} = (1, 0)^T$ elsewhere

Missing information:
unknown BC on the airfoil and the outflow

Setup II - Airfoil - Preliminary results

Velocity recovery error $\|\mathbf{u} - \hat{\mathbf{u}}\|_{H^1(\Omega)^2}$:

m_u \ m_p	0	1	4	9
(0,0)	-	1.0832	0.6862	0.1147
(1,1)	3.5855	0.2781	0.6488	0.0562
(4,4)	0.0906	0.0919	0.0793	0.0119
(9,9)	0.0140	0.0139	0.0127	0.0068

Pressure recovery error $\|p - \hat{p}\|_{L_2(\Omega)}$:

m_u \ m_p	0	1	4	9
(0,0)	-	2.1291	1.24339	0.0830
(1,1)	8.0994	0.5325	1.1970	0.0556
(4,4)	0.1608	0.1611	0.1389	0.0157
(9,9)	0.0168	0.0166	0.0149	0.0082

Setup II - Airfoil - Preliminary results

Drag coefficient recovery error (exact $= 12.553$):

m_u \ m_p	0	1	4	9
(0,0)	-	3.3142	2.0591	0.1117
(1,1)	12.567	0.7148	1.9366	0.0540
(4,4)	0.2203	0.2080	0.1937	0.0101
(9,9)	0.0155	0.0149	0.0124	0.0038

Conclusions

Missing information: Alleviated by measurements;

Hilbertian and Unit Ball Setting: Minimal norm property;

Key step: Approximation of the Reisz representers satisfying fractional diffusion problems;

Practical Near Optimal Algorithm: Approximation of the Chebyshev center; need to know s.

Not discussed: Implementation via saddle point, effect of thresholding in Moore-Penrose, effect of s.

Open question: Optimal measurements.

References:

[Binev, B., Cohen, Dahmen, DeVore, Petrova (original algorithm)];
[B. and Guignard (Practical algorithm for fractional s and Stokes)];
[B., Demlow, and Siktar (interior measurements)].

Acknowledgments: NSF Grants DMS 2110811 - DMS 2409807 and deal.ii.

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Thanks for your attention and Joyeux anniversaire Albert!