Subdivision Schemes-A short review

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Univariate linear Subdivision Schemes

S_a is a **Univariate linear subdivision operator** refining data $f \in \ell_\infty(\mathbb{Z})$

$$(S_a f)(\alpha) = \sum_{\beta \in \mathbb{Z}} a(\alpha - 2\beta) f(\beta), \quad \alpha \in \mathbb{Z}. \tag{1}$$

The refinement by S_a occurs because the value $S_a f(\alpha)$ is attached to the parameter value $\frac{1}{2}\alpha \in \frac{1}{2}\mathbb{Z}$, while $f(\alpha)$ is attached to the parameter value $\alpha \in \mathbb{Z}$.
Here $a \in \ell_\infty(\mathbb{Z})$, has a finite number of non-zero elements. The finite sequence a is termed the **mask** of the subdivision operator.

Note that S_a can operate on sequences of vectors in $\mathbb{R}^d$, operating on each component separately.

Stationary and nonstationary subdivision schemes

A stationary subdivision scheme is based on one subdivision operator S_a. It generates a sequence of sequences in $\ell_\infty(\mathbb{Z})$ from an initial sequence $f^{[0]} \in \ell_\infty(\mathbb{Z})$

$$f^{[k]} = S_a f^{[k-1]} \in \ell_\infty(\mathbb{Z}), \quad k \in \mathbb{N}.$$

A non-stationary subdivision scheme is based on a sequence of subdivision operators $S_{a_0}, S_{a_1}, S_{a_2}, \cdots$, and generates a sequence of sequences in $\ell_\infty(\mathbb{Z})$ from an initial sequence $f^{[0]} \in \ell_\infty(\mathbb{Z})$

$$f^{[k]} = S_{a_{k-1}} f^{[k-1]} \in \ell_\infty(\mathbb{Z}), \quad k \in \mathbb{N}.$$

A subdivision scheme is denoted by $S_{\mathbf{a}}$, with $\mathbf{a} = a$ in the stationary case, while in the nonstationary case $\mathbf{a}$ is the sequence of masks corresponding to the sequence of subdivision operators defining the scheme.

The value $f^{[k]}(\alpha)$ is attached to the parameter value $2^{-k} \cdot \alpha$.

Convergence of subdivision schemes

A subdivision scheme is termed **converging to $C^s(\mathbb{R})$ limit functions**, if for any $f^{[0]} \in \ell_\infty(\mathbb{Z})$ there exists a function $g \in C^s(\mathbb{R})$, nonzero for at least one nonzero $f^{[0]} \in \ell_\infty(\mathbb{Z})$, such that

$$\lim_{k\to\infty} \sup_{\alpha\in\mathbb{Z}} |g(2^{-k}\alpha) - f^{[k]}(\alpha)| = 0. \qquad (2)$$

Note that the limit g is at least continuous. To see this, we replace $f^{[k]}(\alpha)$ in (2) by $P_k(\alpha)$, with P_k, the polygonal line through the points $\{(\alpha, f^{[k]}(\alpha)) \ : \alpha \in \mathbb{Z}\}$, which is a continuous function.

A necessary condition for the convergence of a stationary scheme S_a is

$$\sum_{\alpha\in\mathbb{Z}} a_{2\alpha} = 1, \quad \text{and} \quad \sum_{\alpha\in\mathbb{Z}} a_{2\alpha+1} = 1.$$

Conclusion: The limit of a stationary converging scheme starting from a constant sequence $f^{[0]}_\alpha = c$ *for all* $\alpha \in \mathbb{Z}$ is a constant function having the same value c.

The basic limit function

The **basic limit function** of a converging subdivision scheme, stationary or nonstationary, is the limit obtained by the scheme from $f^{[0]} = \delta$, and is denoted by $\phi_{\mathbf{a}}$. Here $\delta_\alpha = 0$ for $\alpha \in \mathbb{Z} \setminus \{0\}$, and $\delta_0 = 1$. Writing

$$f^{[0]}(\cdot) = \sum_{\alpha \in \mathbb{Z}} f^{[0]}(\alpha)\; delta(\cdot - \alpha),$$

we obtain by the linearity of the subdivision operators, that the limit of a converging subdivision scheme $S_{\mathbf{a}}$, starting from $f^{[0]}$, which we denote by $S_{\mathbf{a}}^{\infty} f^{[0]}$, has the form

$$S_{\mathbf{a}}^{\infty} f^{[0]}(\cdot) = \sum_{\alpha \in \mathbb{Z}} f^{[0]}(\alpha) \phi_{\mathbf{a}}(\cdot - \alpha). \qquad (3)$$

Conclusion: All limits of a converging subdivision scheme have the smoothness of the basic limit function of the scheme.

The first family of converging stationary schemes

B-spline subdivision schemes were developed in the seventy's to evaluate B-spline curves, having the form

$$C_m(t) = \sum_{i=1}^{N} P_i \cdot B_m(t-i).$$

Here $\{P_i\}_{i=1}^{N}$ are given (**control**) points in R^d and $B_m(\cdot)$ is a B-spline of degree m with integer knots and support $[0, m+1]$. B-spline curves are a common tool in CAGD (Computer Aided Geometric Design) for designing curves in Euclidean spaces, since these curves have a similar structure to that of the **control polygon** (the polygonal line through the control points), and are contained in the convex hull of the control points.

The first B-spline subdivision schemes

The mask $a^{[m]}$ of the B-spline scheme of degree m is

$$a^{[m]} = \left(\left(\frac{1}{2}\right)^m \binom{m+1}{i}, \quad i = 0, 1, 2, \ldots, m, m+1 \right).$$

The masks of the first three schemes are:

$$a^{[1]} = \left(\frac{1}{2}, 1, \frac{1}{2}\right), \; a^{[2]} = \left(\frac{1}{4}, \frac{3}{4}, \frac{3}{4}, \frac{1}{4}\right), \; a^{[3]} = \left(\frac{1}{8}, \frac{1}{2}, \frac{3}{4}, \frac{1}{2}, \frac{1}{8}\right).$$

The two refinement rules of the subdivision scheme $S_{a^{[1]}}$ are:

$$f^{[k+1]}(2\alpha) = f^{[k]}(\alpha), \quad f^{[k+1]}(2\alpha+1) = \frac{1}{2} f^{[k]}(\alpha) + \frac{1}{2} f^{[k]}(\alpha+1).$$

For a fixed k, these two rules, are applied for any $\alpha \in \mathbb{Z}$. The refinement is done iteratively for $k = 0, 1, 2, \cdots$

The first B-spline subdivision schemes-continuation

It is easy to see that $\lim_{k\to\infty} S_{a^{[1]}}^k f^{[0]}$ is the polygonal line through the points $\{(\alpha, f^{[0]}(\alpha)),\ \alpha \in \mathbb{Z}\}$.

The two refinement rules of the subdivision scheme $S_{a^{[3]}}$ are:

$$f^{[k+1]}(2\alpha) = \frac{1}{8}(f^{[k]}(\alpha-1) + f^{[k]}(\alpha+1)) + \frac{3}{4}f^{[k]}(\alpha),$$

and

$$f^{[k+1]}(2\alpha+1) = \frac{1}{2}(f^{[k]}(\alpha) + f^{[k]}(\alpha+1)).$$

The limit generated by $S_{a^{[3]}}$ starting from $f^{[0]}$ is the cubic B-spline curve with the control points $\{\alpha, f^{[0]}(\alpha),\ \alpha \in \mathbb{Z}\}$.

The B-spline subdivision schemes, except $S_{a^{[1]}}$, generate curves which do not interpolate the control points.

Interpolatory subdivision schemes

The limit generated by a converging interpolatory subdivision scheme interpolates the initial points $f[0]$. Its mask satisfies $a(2\alpha) = \delta(\alpha)$, corresponding to the refinement rule $f^{[k+1]}(2\alpha) = f^{[k]}(\alpha)$. This refinement rule guarantees that the initial points as well as all the points of each refinement level belong to the limit curve The odd coefficients of the mask define the second refinement rule, which is called **insertion rule**.

The simplest interpolatory subdivision scheme with C^1 limits, which uses symmetric points relative to the interval where a point is inserted is **the four-point scheme**, (Dyn,Gregory, Levin,1987). It has the following insertion rule:

$$f^{[k+1]}(2\alpha+1) = \left(\frac{1}{2}+w\right)\left(f^{[k]}(\alpha)+f^{[k]}(\alpha+1)\right) - w\left(f^{[k]}(\alpha-1)+f^{[k]}(\alpha+2)\right).$$

Here w is a shape parameter. For $|w| \le \frac{1}{2}$ the scheme converges.

Dubuc-Deslauriers interpolatory subdivision schemes (1987)

This family of interpolatory **2n-point schemes**, that we denote by S_{2n}^{DD}, $n = 1, 2, 3, 4, \cdots$ are defined by the following **insertion rule**

$$f^{[k+1]}(2\alpha + 1) = P_{2n-1,\alpha}^{[k]}\left(\alpha + \frac{1}{2}\right)$$

where $P_{2n-1,\alpha}^{[k]}$ is a polynomial of degree $2n - 1$ interpolating the $2n$ symmetric points relative to $[\alpha, \alpha + 1]$

$$(\alpha + i, f^{[k]}(\alpha + i)),\ \ i = -n + 1, -n + 2, \ldots, 0, 1, 2, \ldots, n$$

Note that for $n = 1$ the scheme S_2^{DD} is identical with $S_{a^{[1]}}$, and that the scheme S_4^{DD} is the 4-point scheme with $w = \frac{1}{16}$.
It is easy to observe that if $f^{[0]}$ is sampled from a polynomial of degree $2n - 1$, then the limit curve generated by S_{2n}^{DD} is that polynomial. Thus the space of polynomials of degree $2n - 1$ is generated by S_{2n}^{DD}.

Can ϕ_a be C^∞ and compactly supported?

The smoothness of a converging stationary scheme S_a is bounded by the highest degree of all polynomials of a certain degree that can be generated as limits of the scheme (degree of polynomial generation). The higher this degree is, the support of the mask a is larger, since the mask coefficients must satisfy more "sum rules" to guarantee the generation of higher degree polynomials.

The support of ϕ_a
Since $(S_a \cdot \delta)(\alpha) = a(\alpha)$, then support$(S_a \cdot \delta) = \frac{1}{2}$support$(a)$, and so in case $[0, u]$ is the support of the mask a, then the support of ϕ_a is $[0, u_a]$ with $u_a = \sum_{i=0}^{\infty} \left(\frac{1}{2}\right)^{i+1} \cdot u = u$.
By similar arguments, in case $support(a_i = [0, u_i]$, the support of $\phi_\mathbf{a}$ is $[0, u_\mathbf{a}]$, with $u_\mathbf{a} = \sum_{i=0}^{\infty} \left(\frac{1}{2}\right)^{i+1} \cdot u_i$. Note that the support of $\phi_\mathbf{a}$ can be finite even when u_i are growing
Conclusion : There is no basic limit function of a stationary scheme which is C^∞ and compactly supported.

There is a C^∞ compactly supported $\phi_{\mathbf{a}}$

In a paper by Derfel, Dyn, Levin (1995) it is shown that the basic limit function of the nonstationary scheme $S_{\mathbf{a}}$ with

$$\mathbf{a} = \left(a^{[0]}, a^{[1]}, a^{[2]}, a^{[3]}, \cdots\right),$$

is C^∞, supported on $[0,2]$. It is also observed that this function is identical with the Up-function of Revachev (1971), which solves a probability problem.
Furthermore, the Up-function is related to a sequence of functions $\{\varphi_j,\ j \in \mathbb{N}_0\}$, where φ_j is the basic limit function of a nonstationary scheme with masks $\left(a^{[j+1]}, a^{[j+2]}, a^{[j+3]}, \cdots\right)$, and that these functions satisfy an infinite system of refinement equations

$$\varphi_j(x) = \sum_{k\in\mathbb{Z}} a_k^{[j+1]} \varphi_{j+1}(2x-k),\ j \in \mathbb{N}_0.$$

On the generation of C^∞ compactly supported $\phi_{\mathbf{a}}$

In a joint paper by Albert Cohen and Nira Dyn (1997), a theoretical study of nonstationary schemes with basic limit functions that are C^∞ and compactly supported is presented.

Two examples are investigated: The nonstationary scheme consisting of the subdivision operators

$$(S_2^{DD}, S_4^{DD}, S_6^{DD}, \cdots),$$

which is interpolatory at each refinement level and therefore interpolatory, and a nonstationary scheme which generates C^∞ compactly supported orthonormal wavelets, based on the family of Daubechies' filters for generating orthonormal wavelets of growing smoothness.